



# INTRODUCTION TO BOOLEAN ALGEBRA

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Developed for the Azera Group

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## Why Study Boolean Algebra?

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- Understand the relationship between Boolean logic and digital computer circuits.
- Learn how to design simple logic circuits.
- Understand how digital circuits work together to form complex computer systems.



## DEFINITION

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- Boolean algebra is a mathematical system for the manipulation of variables that can have one of two values.
  - In formal logic, these values are “true” and “false.”
  - In digital systems, these values are “on” and “off,” 1 and 0, or “high” and “low.”
- Boolean expressions are created by performing operations on Boolean variables.
  - Common Boolean operators include AND, OR, and NOT.

## Boolean Operators and Truth Tables

- A Boolean operator can be completely described using a truth table.
- The truth table for the Boolean operators AND and OR are shown at the right.
- The AND operator is also known as a Boolean product. The OR operator is the Boolean sum.

X AND Y

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

X OR Y

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

## Boolean Operators and Truth Tables: Part Two

- The truth table for the Boolean NOT operator is shown at the right.
- The NOT operation is most often designated by an overbar. It is sometimes indicated by a prime mark ( ' ) or an “elbow” ( $\neg$ ).

x	$\bar{x}$
0	1
1	0

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## Boolean Functions

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- A Boolean function has:
  - At least one Boolean variable,
  - At least one Boolean operator, and
  - At least one input from the set  $\{0,1\}$ .
- It produces an output that is also a member of the set  $\{0,1\}$ .

# Boolean Functions and Truth Tables

- The truth table for the Boolean function:

$$F(x, y, z) = x\bar{z} + y$$

is shown at the right.

- To make evaluation of the Boolean function easier, the truth table contains extra (shaded) columns to hold evaluations of subparts of the function.

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	$\bar{z}$	$x\bar{z}$	$x\bar{z} + y$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1

# Boolean Operations Rules of Precedence

- As with common arithmetic, Boolean operations have rules of precedence.
- The NOT operator has highest priority, followed by AND and then OR.
- This is how we chose the (shaded) function subparts in our table.

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	$\bar{z}$	$x\bar{z}$	$x\bar{z} + y$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	0	1





## Computer Representations of Boolean Functions

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- Digital computers contain circuits that implement Boolean functions.
- The simpler that we can make a Boolean function, the smaller the circuit that will result.
  - Simpler circuits are cheaper to build, consume less power, and run faster than complex circuits.
- With this in mind, we always want to reduce our Boolean functions to their simplest form.
- There are a number of Boolean identities that help us to do this.

## BOOLEAN IDENTITIES

- Most Boolean identities have an AND (product) form as well as an OR (sum) form. We give our identities using both forms. Our first group is rather intuitive:

Identity Name	AND Form	OR Form
Identity Law	$1x = x$	$0 + x = x$
Null Law	$0x = 0$	$1 + x = 1$
Idempotent Law	$xx = x$	$x + x = x$
Inverse Law	$x\bar{x} = 0$	$x + \bar{x} = 1$

# ALGEBRAIC RULES PART ONE

- Our second group of Boolean identities should be familiar to you from your study of algebra:

Identity Name	AND Form	OR Form
Commutative Law	$xy = yx$	$x+y = y+x$
Associative Law	$(xy)z = x(yz)$	$(x+y)+z = x+(y+z)$
Distributive Law	$x+yz = (x+y)(x+z)$	$x(y+z) = xy+xz$

## ALGEBRAIC RULES PART TWO

- Our last group of Boolean identities are perhaps the most useful.
- If you have studied set theory or formal logic, these laws are also familiar to you.

Identity Name	AND Form	OR Form
Absorption Law	$x(x+y) = x$	$x + xy = x$
DeMorgan's Law	$\overline{(xy)} = \bar{x} + \bar{y}$	$\overline{(x+y)} = \bar{x}\bar{y}$
Double Complement Law	$\overline{(\bar{x})} = x$	

# BOOLEAN FUNCTION SIMPLIFICATION

- We can use Boolean identities to simplify the function:  $F(X, Y, Z) = (X + Y)(X + \bar{Y})(\overline{XZ})$  as follows:

$(X + Y)(X + \bar{Y})(\overline{XZ})$	Idempotent Law (Rewriting)
$(X + Y)(X + \bar{Y})(\bar{X} + Z)$	DeMorgan's Law
$(XX + X\bar{Y} + XY + Y\bar{Y})(\bar{X} + Z)$	Distributive Law
$((X + Y\bar{Y}) + X(Y + \bar{Y}))(\bar{X} + Z)$	Commutative & Distributive Laws
$((X + 0) + X(1))(\bar{X} + Z)$	Inverse Law
$X(\bar{X} + Z)$	Idempotent Law
$X\bar{X} + XZ$	Distributive Law
$0 + XZ$	Inverse Law
$XZ$	Idempotent Law

## DEMORGAN'S LAW: PART ONE

- Sometimes it is more economical to build a circuit using the complement of a function (and complementing its result) than it is to implement the function directly.
- DeMorgan's law provides an easy way of finding the complement of a Boolean function.
- Recall DeMorgan's law states:

$$\overline{(xy)} = \bar{x} + \bar{y} \quad \text{and} \quad \overline{(x+y)} = \bar{x}\bar{y}$$

## DEMORGAN'S LAW: PART TWO

- DeMorgan's law can be extended to any number of variables.
- Replace each variable by its complement and change all ANDs to ORs and all ORs to ANDs.
- Thus, we find the complement of:

$$F(X, Y, Z) = (XY) + (\bar{X}Z) + (Y\bar{Z})$$

is:

$$\begin{aligned}\bar{F}(X, Y, Z) &= \overline{(XY) + (\bar{X}Z) + (Y\bar{Z})} \\ &= \overline{(XY)} \overline{(\bar{X}Z)} \overline{(Y\bar{Z})} \\ &= (\bar{X} + \bar{Y})(X + \bar{Z})(\bar{Y} + Z)\end{aligned}$$



## LOGICAL EQUIVALENTS

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- Through our exercises in simplifying Boolean expressions, we see that there are numerous ways of stating the same Boolean expression.
  - These “synonymous” forms are *logically equivalent*.
  - Logically equivalent expressions have identical truth tables.
- In order to eliminate as much confusion as possible, designers express Boolean functions in *standardized or canonical* form.



# CANONICAL FORMS OF BOOLEAN EXPRESSIONS: PART ONE

- There are two canonical forms for Boolean expressions: sum-of-products and product-of-sums.
  - The Boolean product is the AND operation and the Boolean sum is the OR operation.
- In the sum-of-products form, AND-ed variables are OR-ed together.
  - For example:
- In the product-of-sums form, OR-ed variables are AND-ed together:
  - For example:

$$F(x, y, z) = xy + xz + yz$$

$$F(x, y, z) = (x+y)(x+z)(y+z)$$

## CANONICAL FORMS OF BOOLEAN EXPRESSIONS: PART TWO

- To convert a function to sum-of-products form use its truth table.
- We are interested in the values of the variables that make the function true (=1).
- Using the truth table, we list the values of the variables that result in a true function value.
- Each group of variables is then OR-ed together.

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	$x\bar{z} + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

# CANONICAL FORMS OF BOOLEAN EXPRESSIONS: PART THREE

- The sum-of-products form for our function is:

$$F(x, y, z) = \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}\bar{z} + xy\bar{z} + xyz$$

**We note that this function is not in simplest terms. Our aim is only to rewrite our function in canonical sum-of-products form.**

$$F(x, y, z) = x\bar{z} + y$$

x	y	z	$x\bar{z} + y$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

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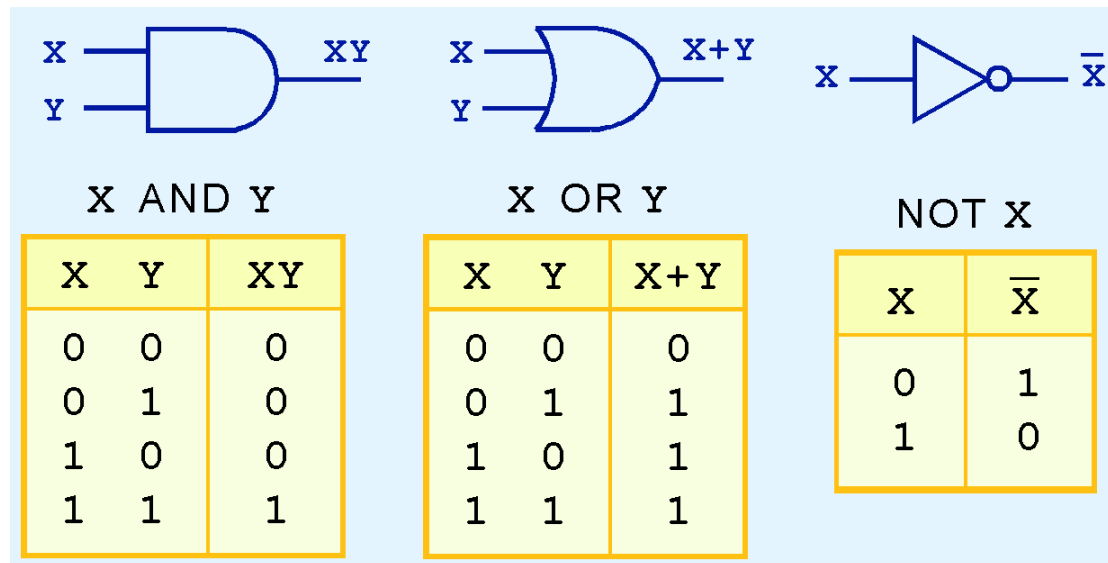
## IMPLEMENTATION OF BOOLEAN FUNCTIONS

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- We have looked at Boolean functions in abstract terms.
- In this section, we see that Boolean functions are implemented in digital computer circuits called gates.
- A gate is an electronic device that produces a result based on two or more input values.
  - In reality, gates consist of one to six transistors, but digital designers think of them as a single unit.
  - Integrated circuits contain collections of gates suited to a particular purpose.

# THREE BASIC LOGIC GATES

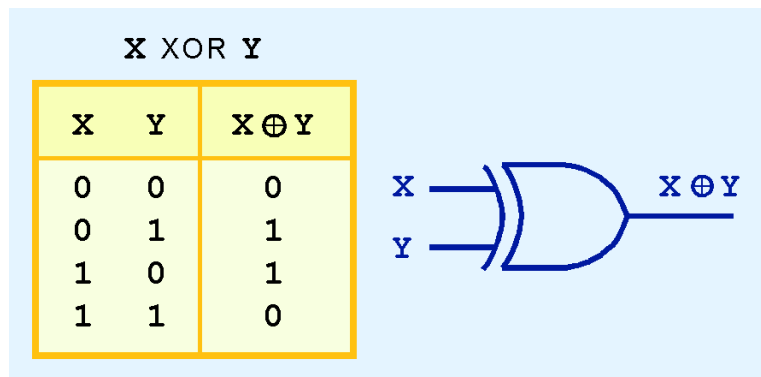
- The three simplest gates are the AND, OR, and NOT gates.



- They correspond directly to their respective Boolean operations, as you can see by their truth tables.

## EXCLUSIVE OR GATE XOR

- Another very useful gate is the exclusive OR (XOR) gate.
- The output of the XOR operation is true only when the values of the inputs differ.



**Note the special symbol  $\oplus$  for the XOR operation.**

# NAND AND NOR GATES: PART ONE

- NAND and NOR are two very important gates. Their symbols and truth tables are shown at the right.

X NAND Y

X	Y	X NAND Y
0	0	1
0	1	1
1	0	1
1	1	0



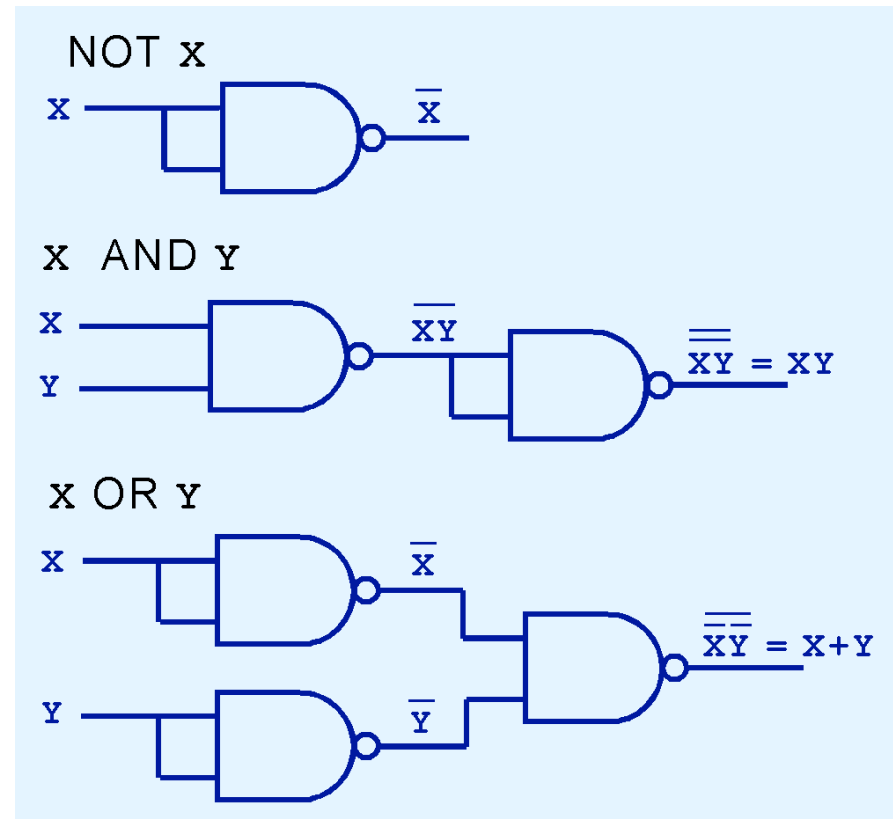
X NOR Y

X	Y	X NOR Y
0	0	1
0	1	0
1	0	0
1	1	0



## NAND AND NOR GATES: PART TWO

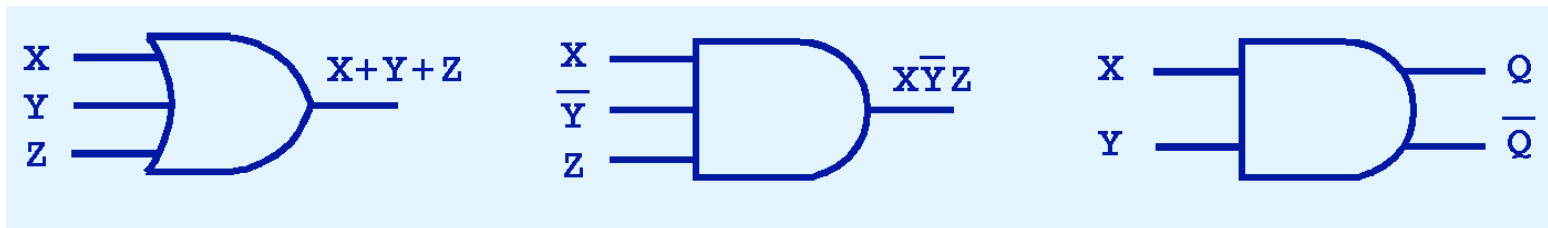
- NAND and NOR are known as *universal gates* because they are inexpensive to manufacture and any Boolean function can be constructed using only NAND or only NOR gates.





## FEATURES OF LOGIC GATES: PART ONE

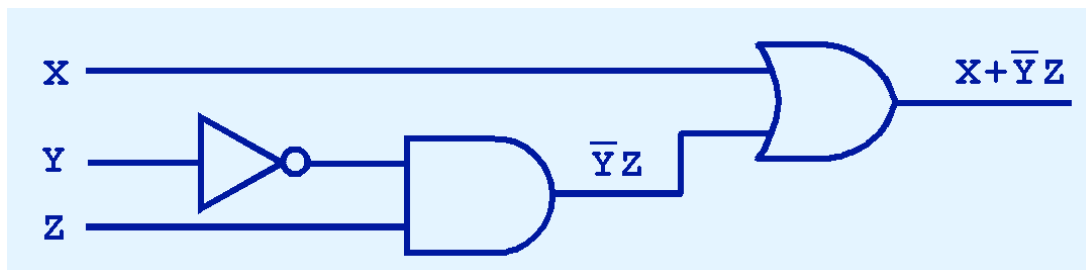
- Gates can have multiple inputs and more than one output.
  - A second output can be provided for the complement of the operation.
  - We'll see more of this later.



## FEATURES OF LOGIC GATES: PART TWO

- The main thing to remember is that combinations of gates implement Boolean functions.
- The circuit below implements the Boolean function:

$$F(X, Y, Z) = X + \bar{Y}Z$$



**We simplify our Boolean expressions so that we can create simpler circuits.**

## COMBINATIONAL LOGIC CIRCUIT GATES: ONE

- We have designed a circuit that implements the Boolean function:

$$F(X, Y, Z) = X + \bar{Y}Z$$

- This circuit is an example of a *combinational logic* circuit.
- Combinational logic circuits produce a specified output (almost) at the instant when input values are applied.
  - In a later section, we will explore circuits where this is not the case.

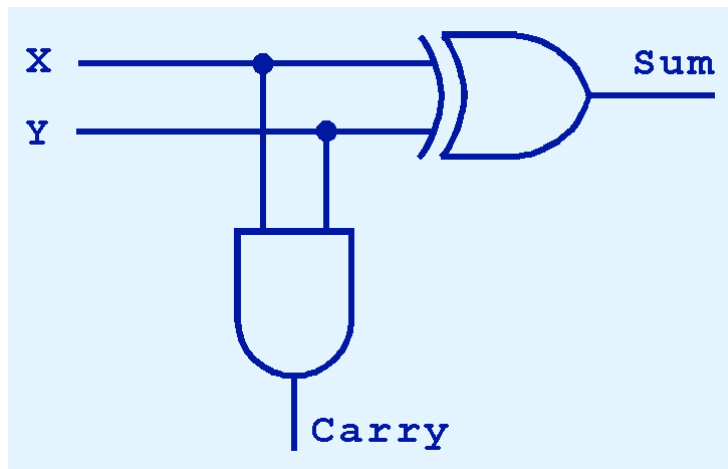
## LOGIC CIRCUIT: HALF ADDER PART ONE

- Combinational logic circuits give us many useful devices.
- One of the simplest is the *half adder*, which finds the sum of two bits.
- We can gain some insight as to the construction of a half adder by looking at its truth table, shown at the right.

Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

## LOGIC CIRCUIT: HALF ADDER PART TWO

- As we see, the sum can be found using the XOR operation and the carry using the AND operation.



Inputs		Outputs	
X	Y	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

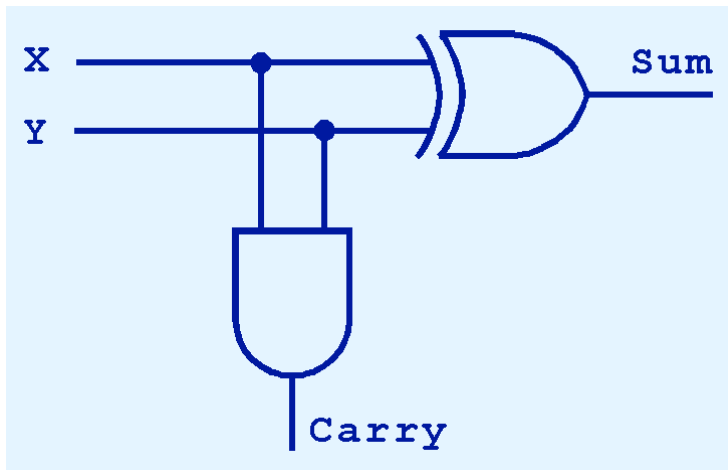
## LOGIC CIRCUIT: HALF ADDER PART THREE

- We can change our half adder into a full adder by including gates for processing the carry bit.
- The truth table for a full adder is shown at the right.

Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

# LOGIC CIRCUIT: FULL ADDER PART ONE

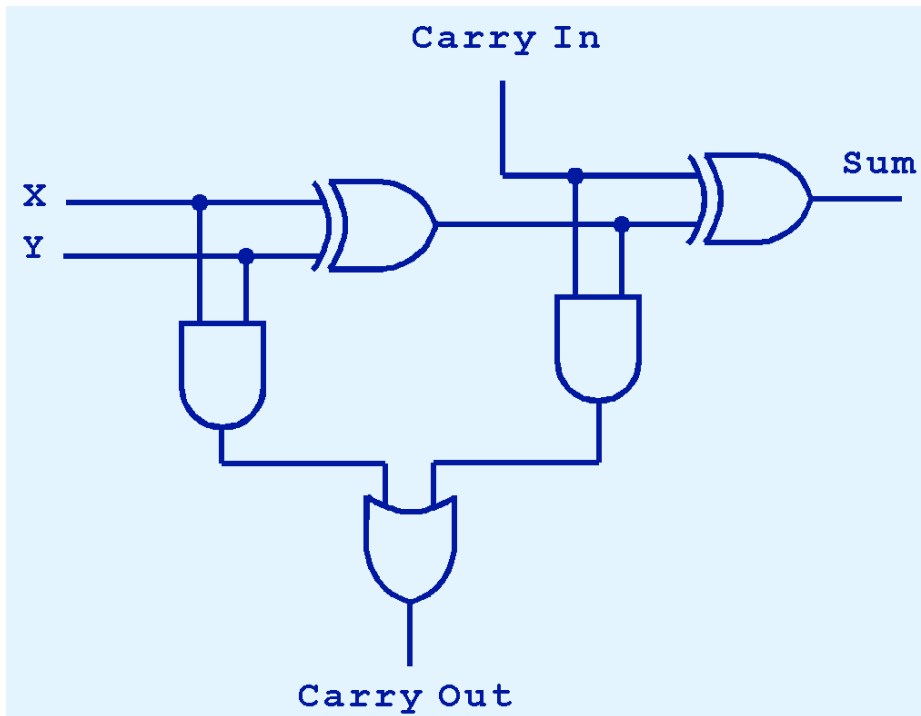
- How can we change the half adder shown below to make it a full adder?



Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

# LOGIC CIRCUIT: FULL ADDER PART TWO

- Here's our completed full adder.

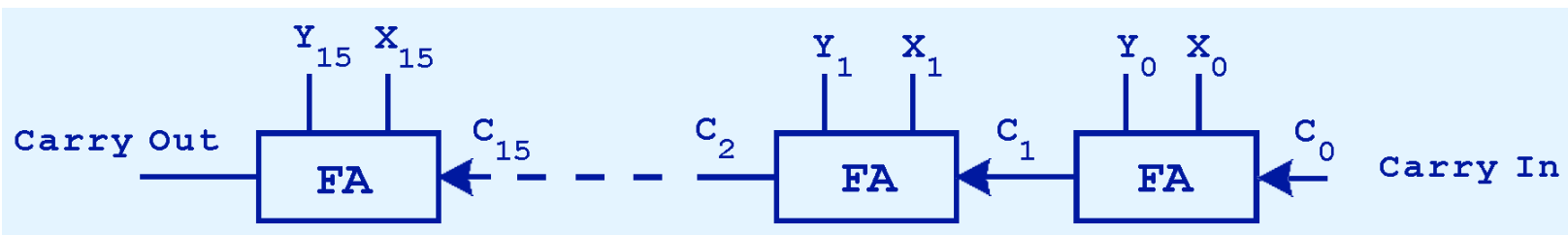


Inputs			Outputs	
X	Y	Carry In	Sum	Carry Out
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



## LOGIC CIRCUIT: RIPPLE-CARRY ADDER

- Just as we combined half adders to make a full adder, full adders can be connected in series.
- The carry bit “ripples” from one adder to the next; hence, this configuration is called a *ripple-carry adder*.



**Today's systems employ more efficient adders.**

## LOGIC CIRCUIT: DECODER: PART ONE

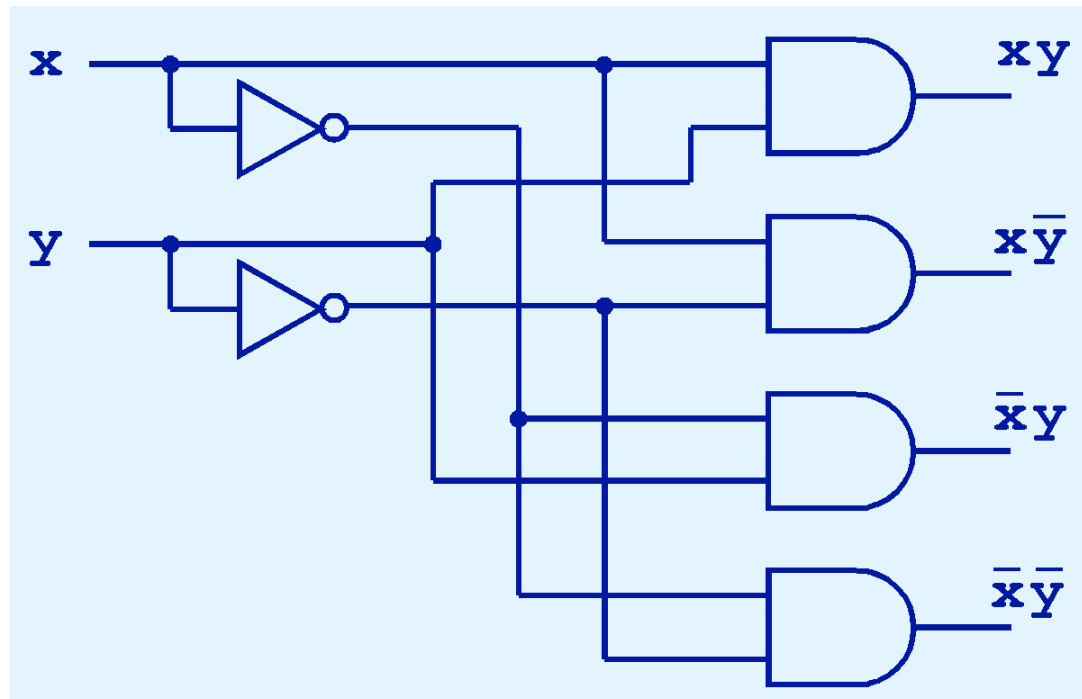
- Decoders are another important type of combinational circuit.
- Among other things, they are useful in selecting a memory location according a binary value placed on the address lines of a memory bus.
- Address decoders with  $n$  inputs can select any of  $2^n$  locations.

**This is a block diagram for a decoder.**



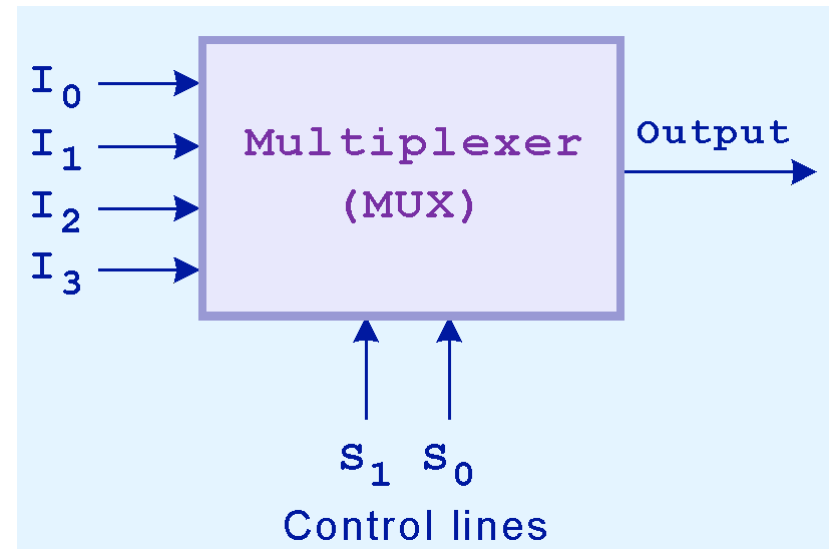
## LOGIC CIRCUIT: DECODER: PART TWO

- This is what a 2-to-4 decoder looks like on the inside.



## LOGIC CIRCUIT: MULTIPLEXER: PART ONE

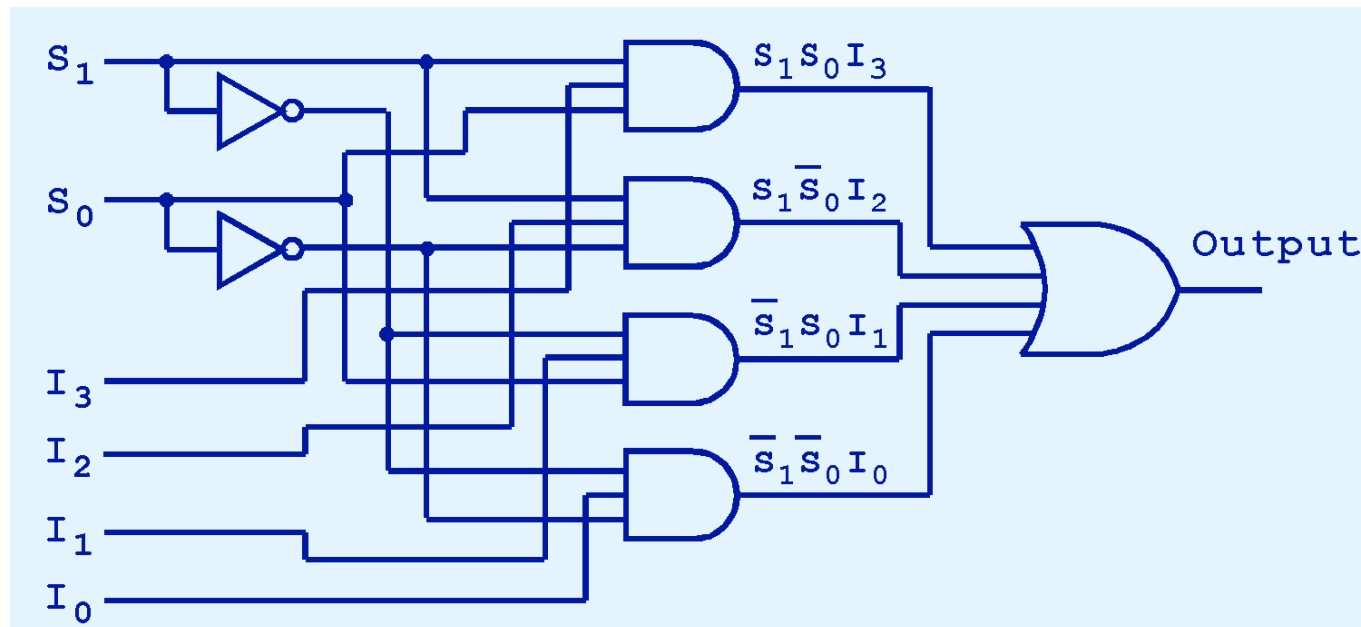
- A multiplexer does just the opposite of a decoder.
- It selects a single output from several inputs.
- The particular input chosen for output is determined by the value of the multiplexer's control lines.
- To be able to select among  $n$  inputs,  $\log_2 n$  control lines are needed.



**This is a block diagram for a multiplexer.**

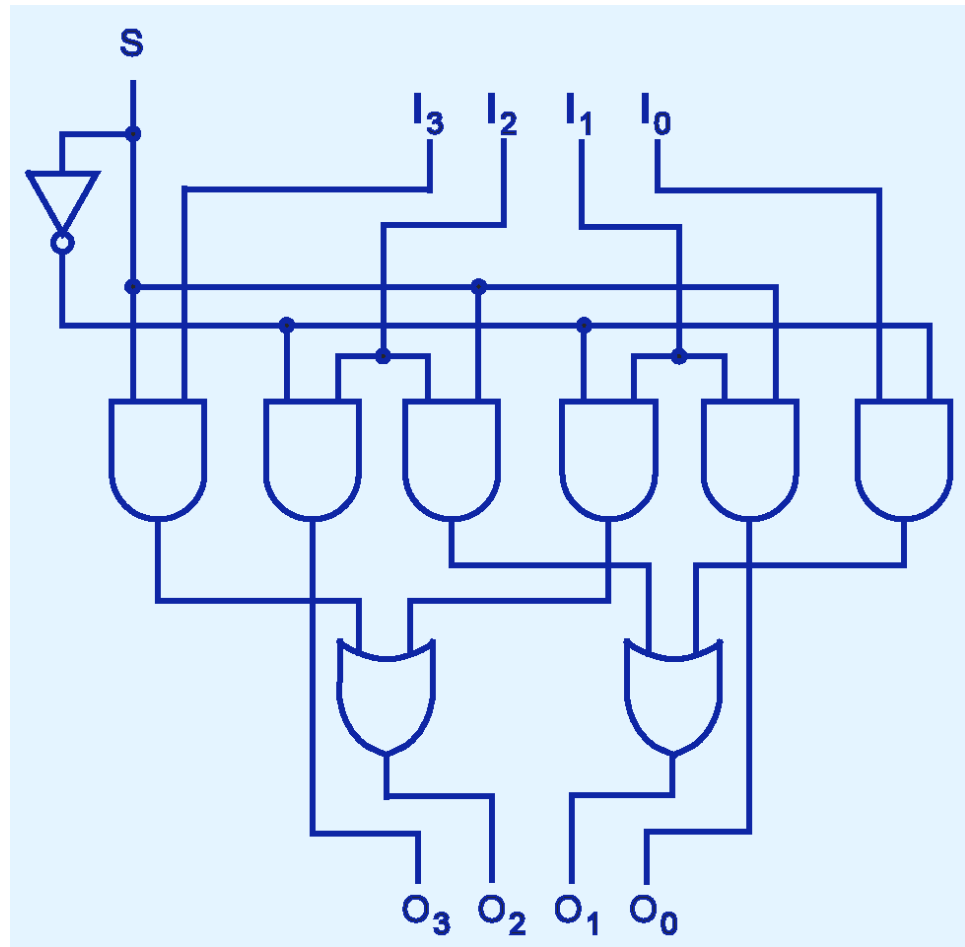
## LOGIC CIRCUIT: MULTIPLEXER: PART TWO

- This is what a 4-to-1 multiplexer looks like on the inside.



# LOGIC CIRCUIT: SHIFTER

- This shifter moves the bits of a nibble one position to the left or right.





## SEQUENTIAL LOGIC CIRCUITS: PART ONE

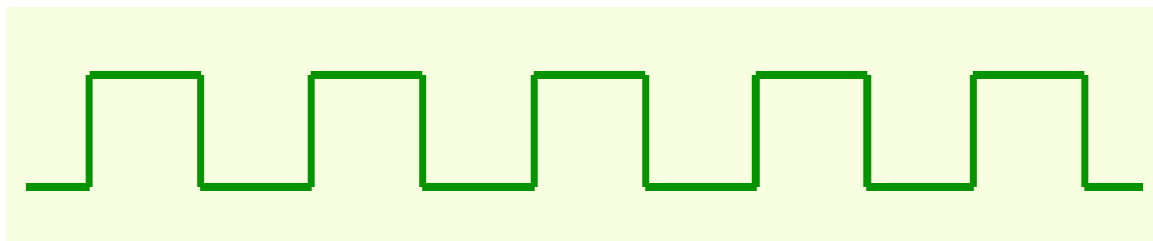
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- Combinational logic circuits are perfect for situations when we require the immediate application of a Boolean function to a set of inputs.
- There are other times, however, when we need a circuit to change its value with consideration to its current state as well as its inputs.
  - These circuits have to “remember” their current state.
- *Sequential logic circuits* provide this functionality for us.

## SEQUENTIAL LOGIC CIRCUITS: PART TWO

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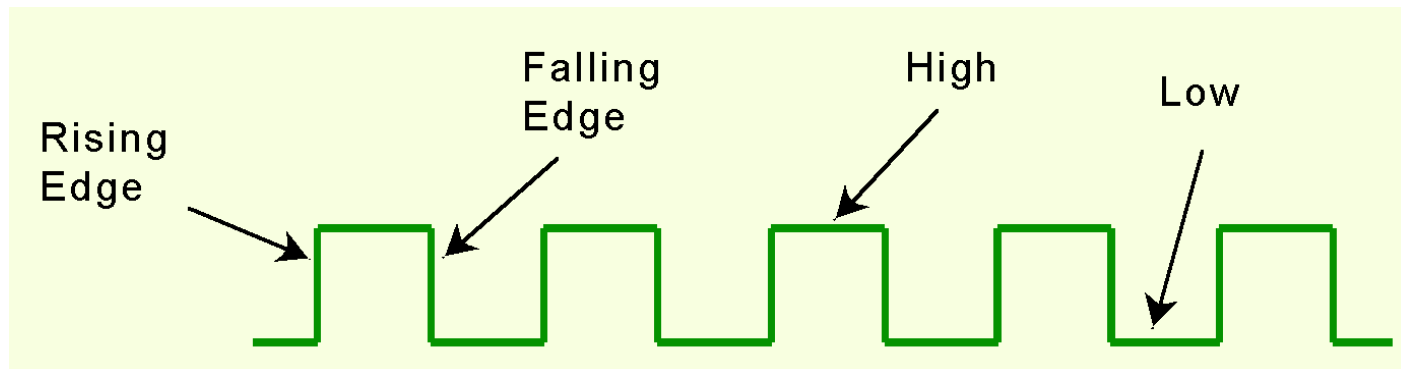
- As the name implies, sequential logic circuits require a means by which events can be sequenced.
- State changes are controlled by clocks.
  - A “clock” is a special circuit that sends electrical pulses through a circuit.
- Clocks produce electrical waveforms such as the one shown below.





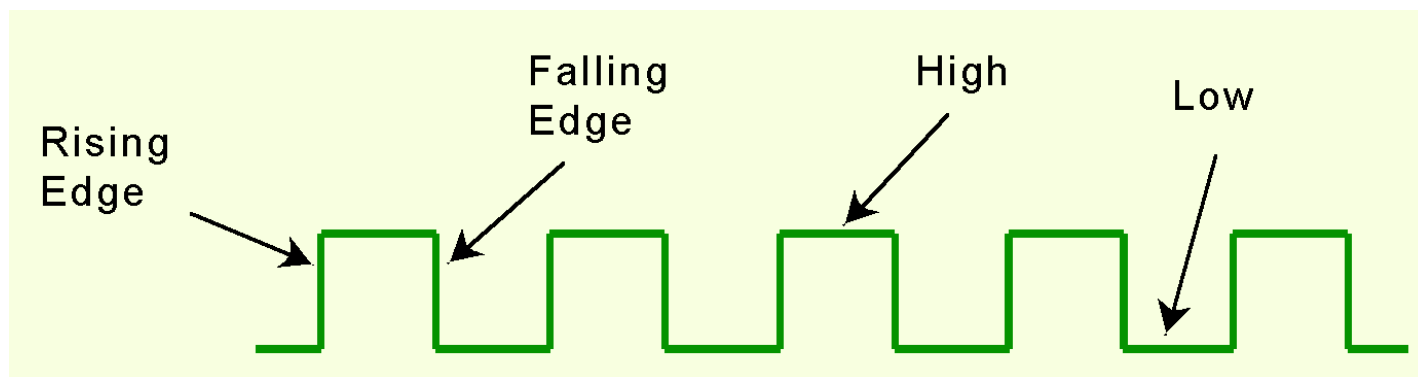
## SEQUENTIAL LOGIC CIRCUITS: PART THREE

- State changes occur in sequential circuits only when the clock ticks.
- Circuits can change state on the rising edge, falling edge, or when the clock pulse reaches its highest voltage.



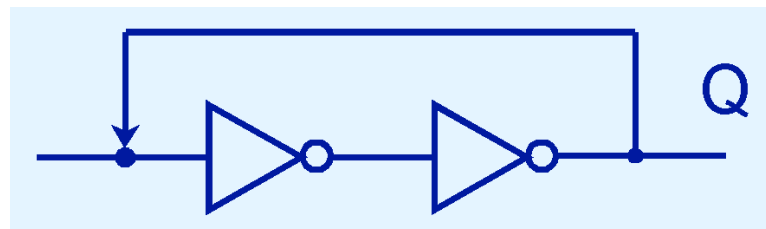
## SEQUENTIAL LOGIC CIRCUITS: PART FOUR

- Circuits that change state on the rising edge, or falling edge of the clock pulse are called *edge-triggered*.
- *Level-triggered circuits* change state when the clock voltage reaches its highest or lowest level.



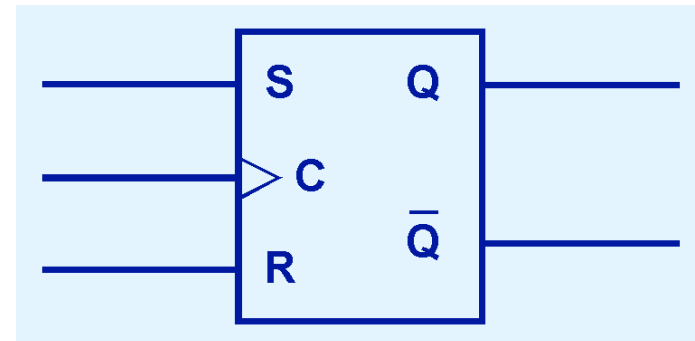
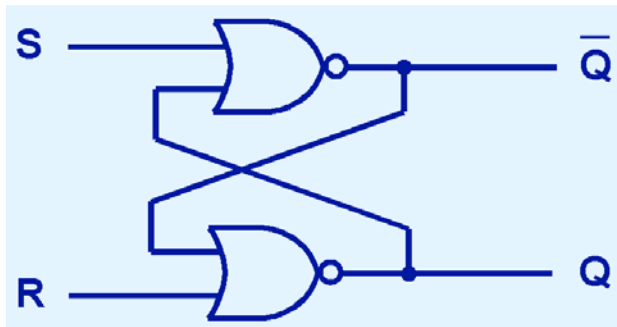
## LOGIC CIRCUITS: FEEDBACK PART ONE

- To retain their state values, sequential circuits rely on *feedback*.
- Feedback in digital circuits occurs when an output is looped back to the input.
- A simple example of this concept is shown below.
  - If Q is 0 it will always be 0, if it is 1, it will always be 1. Why?



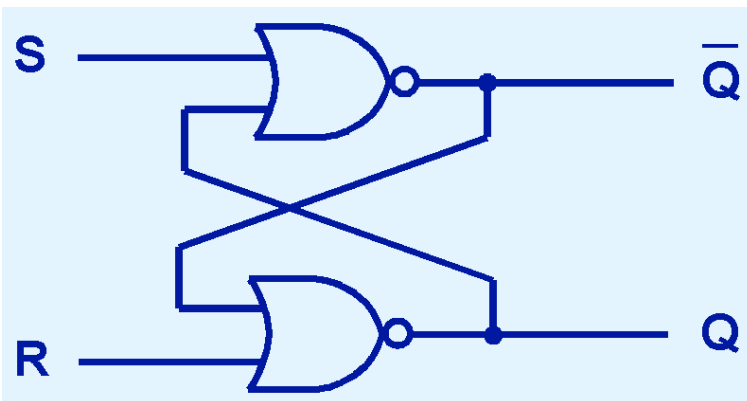
## LOGIC CIRCUITS: SET/RESET FLIP-FLOP PART ONE

- You can see how feedback works by examining the most basic sequential logic components, the SR flip-flop.
  - The “SR” stands for set/reset.
- The internals of an SR flip-flop are shown below, along with its block diagram.



## LOGIC CIRCUITS: SET/RESET FLIP-FLOP PART TWO

- The behavior of an SR flip-flop is described by a characteristic table.
- $Q(t)$  means the value of the output at time  $t$ .  
 $Q(t+1)$  is the value of  $Q$  after the next clock pulse.



S	R	$Q(t+1)$
0	0	$Q(t)$ (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	undefined

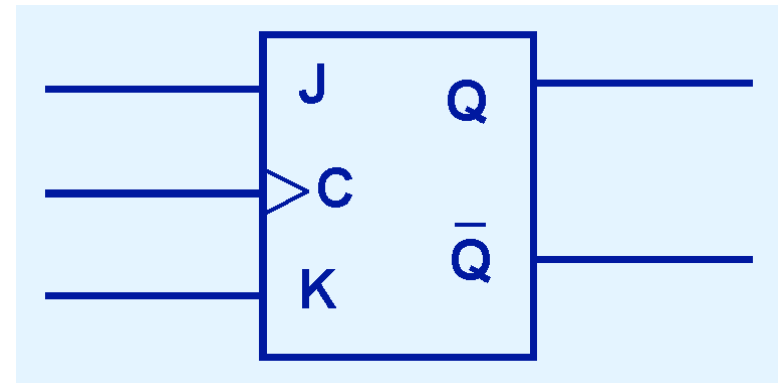
## LOGIC CIRCUITS: SET/RESET FLIP-FLOP PART THREE

- The SR flip-flop actually has three inputs: S, R, and its current output, Q.
- Thus, we can construct a truth table for this circuit, as shown at the right.
- Notice the two undefined values. When both S and R are 1, the SR flip-flop is unstable.

Present State			Next State
S	R	Q(t)	Q(t+1)
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	undefined
1	1	1	undefined

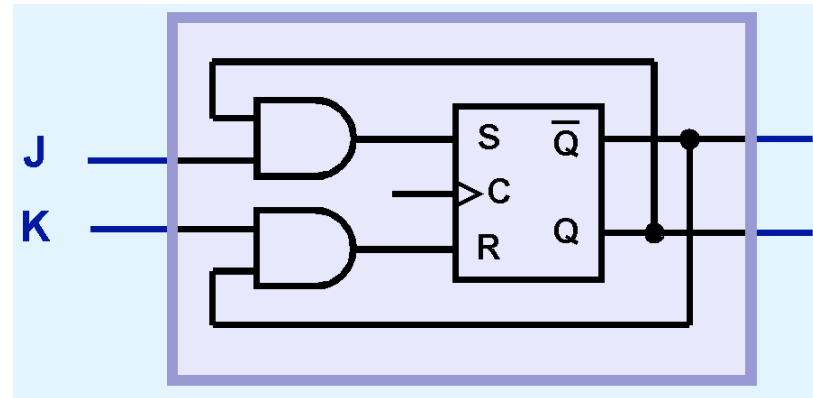
## LOGIC CIRCUITS: JK FLIP-FLOP PART ONE

- If we can be sure that the inputs to an SR flip-flop will never both be 1, we will never have an unstable circuit. This may not always be the case.
  - The SR flip-flop can be modified to provide a stable state when both inputs are 1.
- 
- This modified flip-flop is called a JK flip-flop, shown at the right.



## LOGIC CIRCUITS: JK FLIP-FLOP PART TWO

- At the right, we see how an SR flip-flop can be modified to create a JK flip-flop.
- The characteristic table indicates that the flip-flop is stable for all inputs.

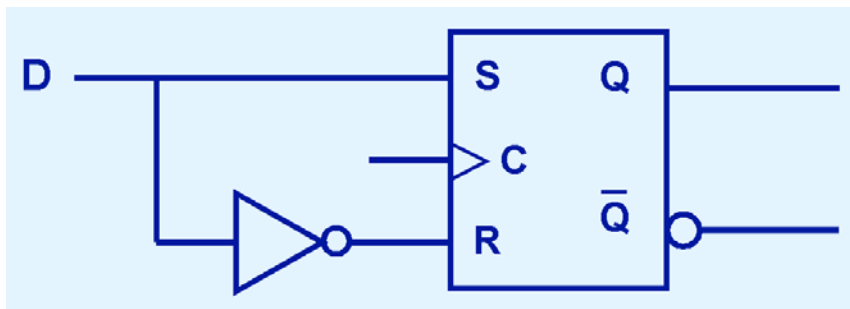


J	K	$Q(t+1)$
0	0	$Q(t)$ (no change)
0	1	0 (reset to 0)
1	0	1 (set to 1)
1	1	$\bar{Q}(t)$



## LOGIC CIRCUITS: D FLIP-FLOP PART ONE

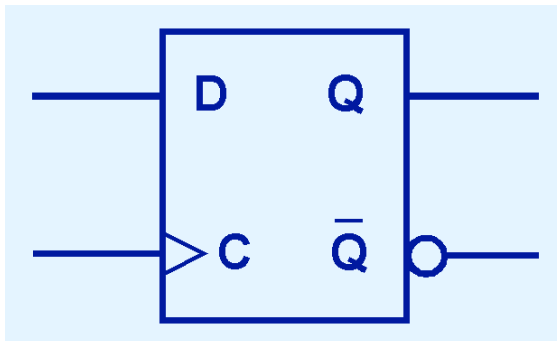
- Another modification of the SR flip-flop is the D flip-flop, shown below with its characteristic table.
- You will notice that the output of the flip-flop remains the same during subsequent clock pulses. The output changes only when the value of D changes.



D	Q (t+1)
0	0
1	1

## LOGIC CIRCUITS: D FLIP-FLOP PART TWO

- The D flip-flop is the fundamental circuit of computer memory.
  - D flip-flops are usually illustrated using the block diagram shown below.
- The characteristic table for the D flip-flop is shown at the right.



D	Q (t+1)
0	0
1	1



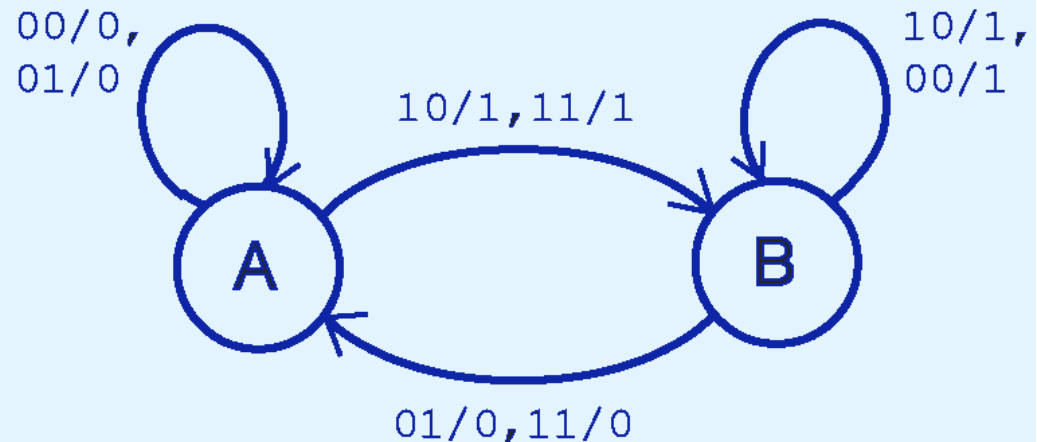
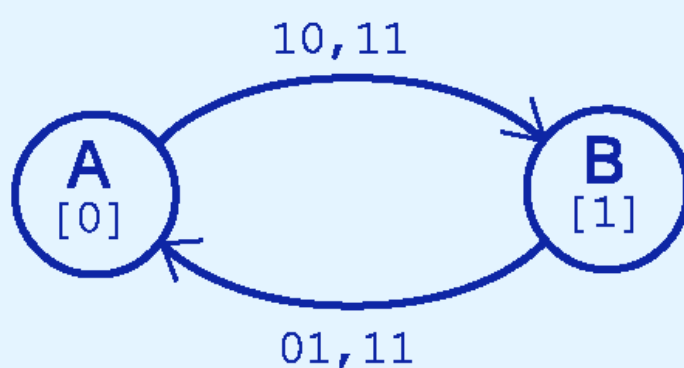
## FINITE STATE MACHINES: PART ONE

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- The behavior of sequential circuits can be expressed using characteristic tables or finite state machines (FSMs).
  - FSMs consist of a set of nodes that hold the states of the machine and a set of arcs that connect the states.
- Moore and Mealy machines are two types of FSMs that are equivalent.
  - They differ only in how they express the outputs of the machine.
- Moore machines place outputs on each node, while Mealy machines present their outputs on the transitions.

## FINITE STATE MACHINES: PART TWO

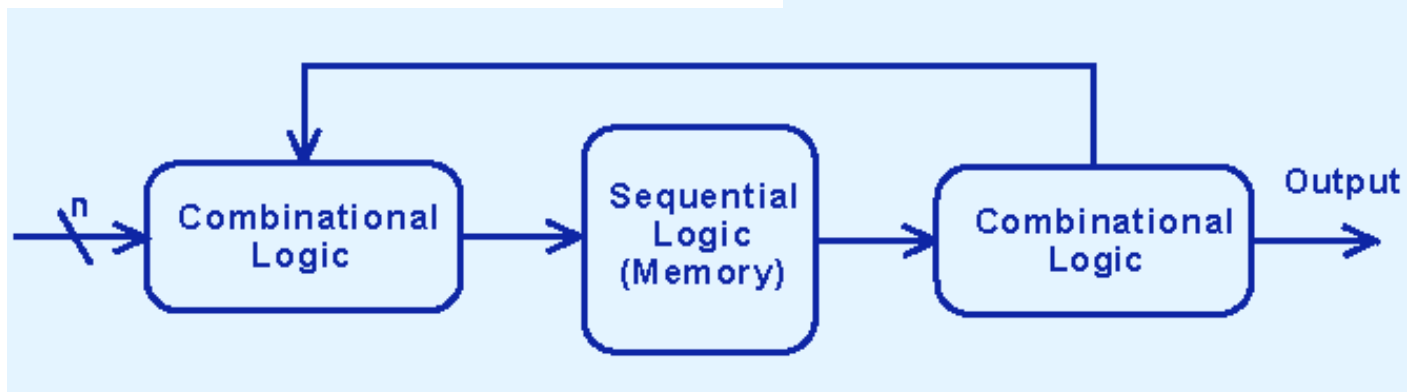
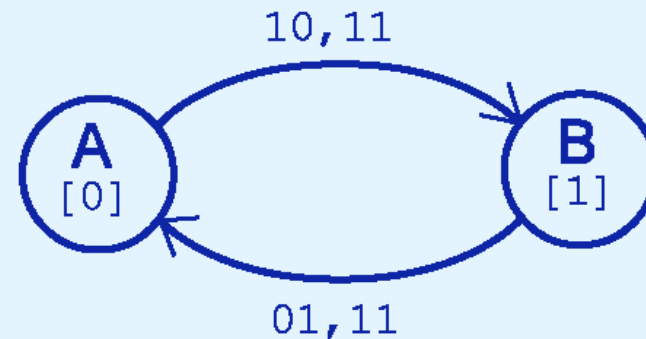
- The behavior of a JK flop-flop is depicted below by a Moore machine (left) and a Mealy machine (right).



## FINITE STATE MACHINES: MOORE MACHINE

- Although the behavior of Moore and Mealy machines is identical, their implementations differ.

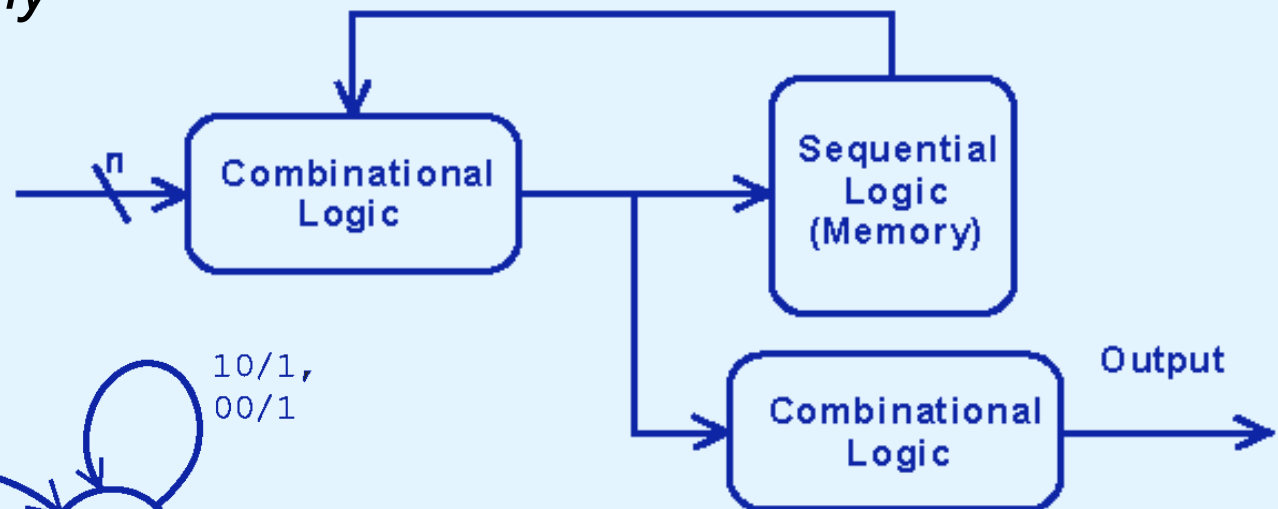
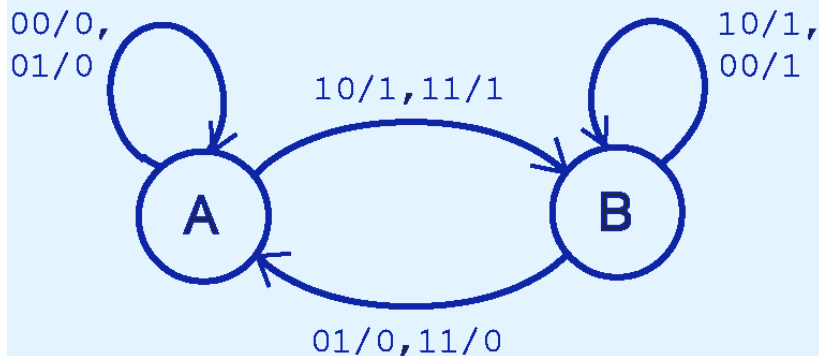
This is our Moore machine.



## FINITE STATE MACHINES: MEALY MACHINE

- Although the behavior of Moore and Mealy machines is identical, their implementations differ.

*This is our Mealy machine.*



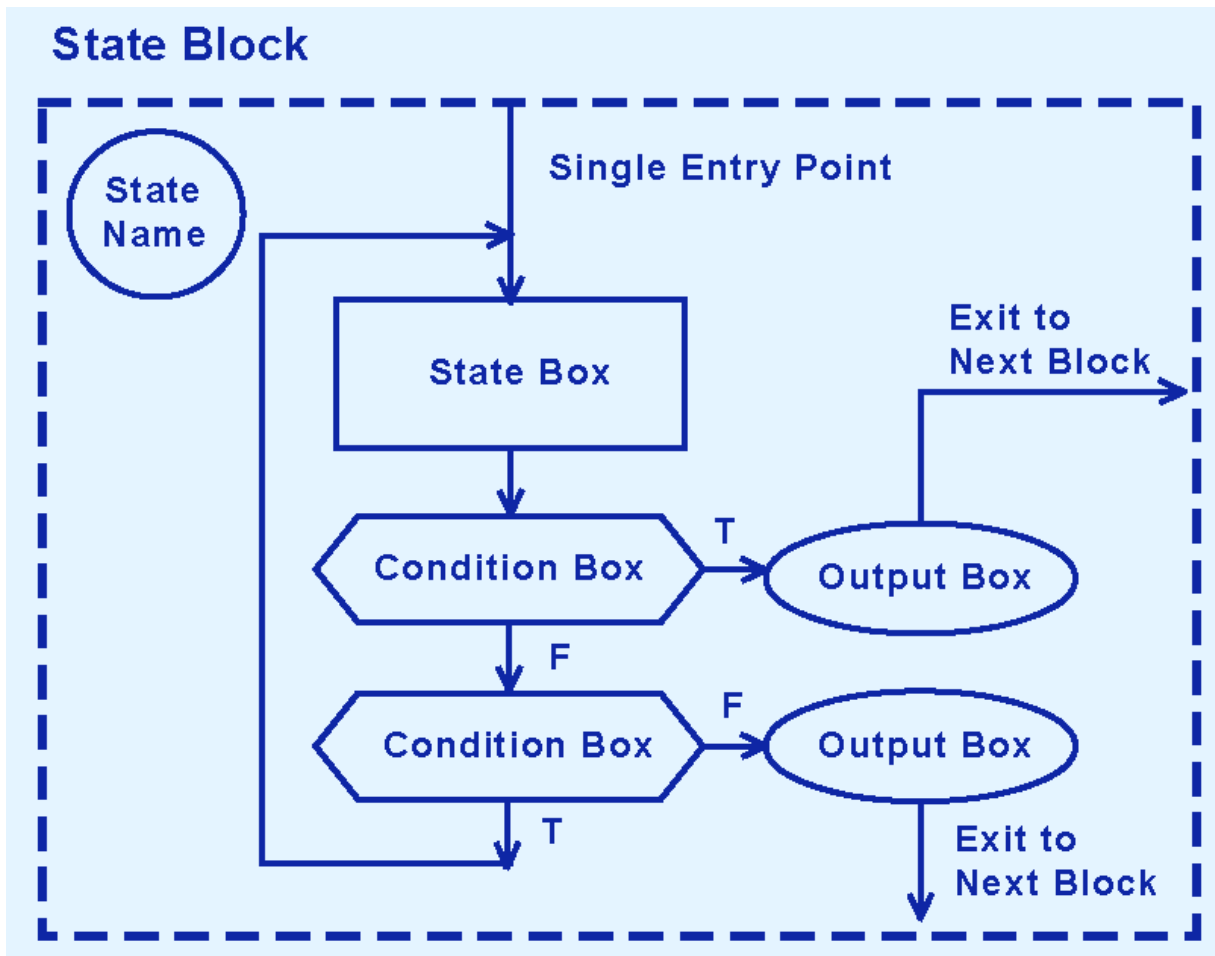


## ALGORITHMIC STATE MACHINES: PART ONE

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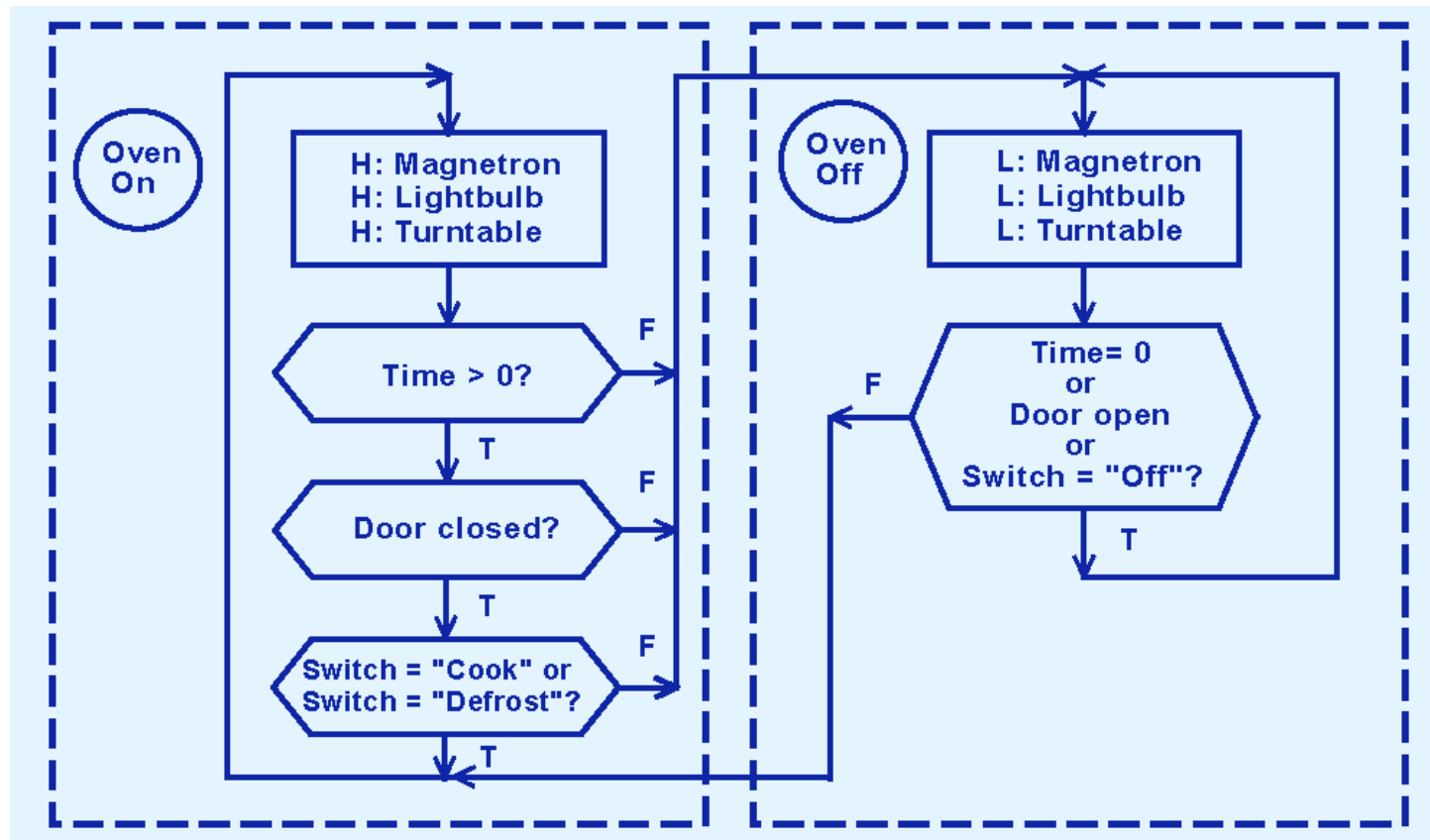
- It is difficult to express the complexities of actual implementations using only Moore and Mealy machines.
  - For one thing, they do not address the intricacies of timing very well.
  - Secondly, it is often the case that an interaction of numerous signals is required to advance a machine from one state to the next.
- For these reasons, Christopher Clare invented the algorithmic state machine (ASM).

## ALGORITHMIC STATE MACHINE: PART TWO





# THE ASM for a microwave oven.





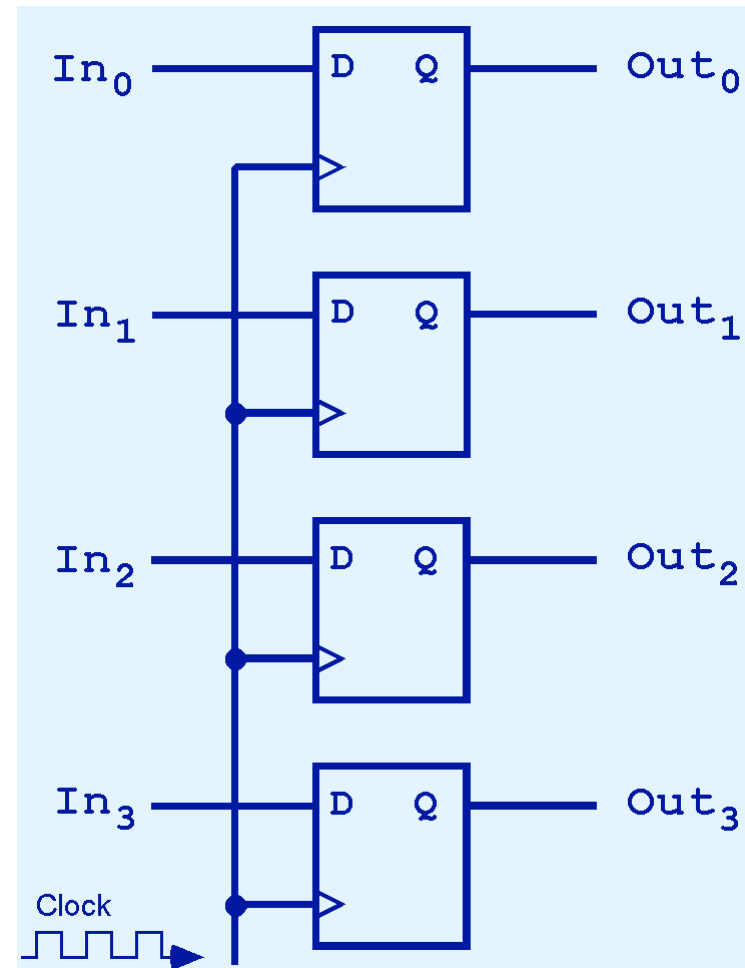
## SEQUENTIAL CIRCUITS: PART ONE

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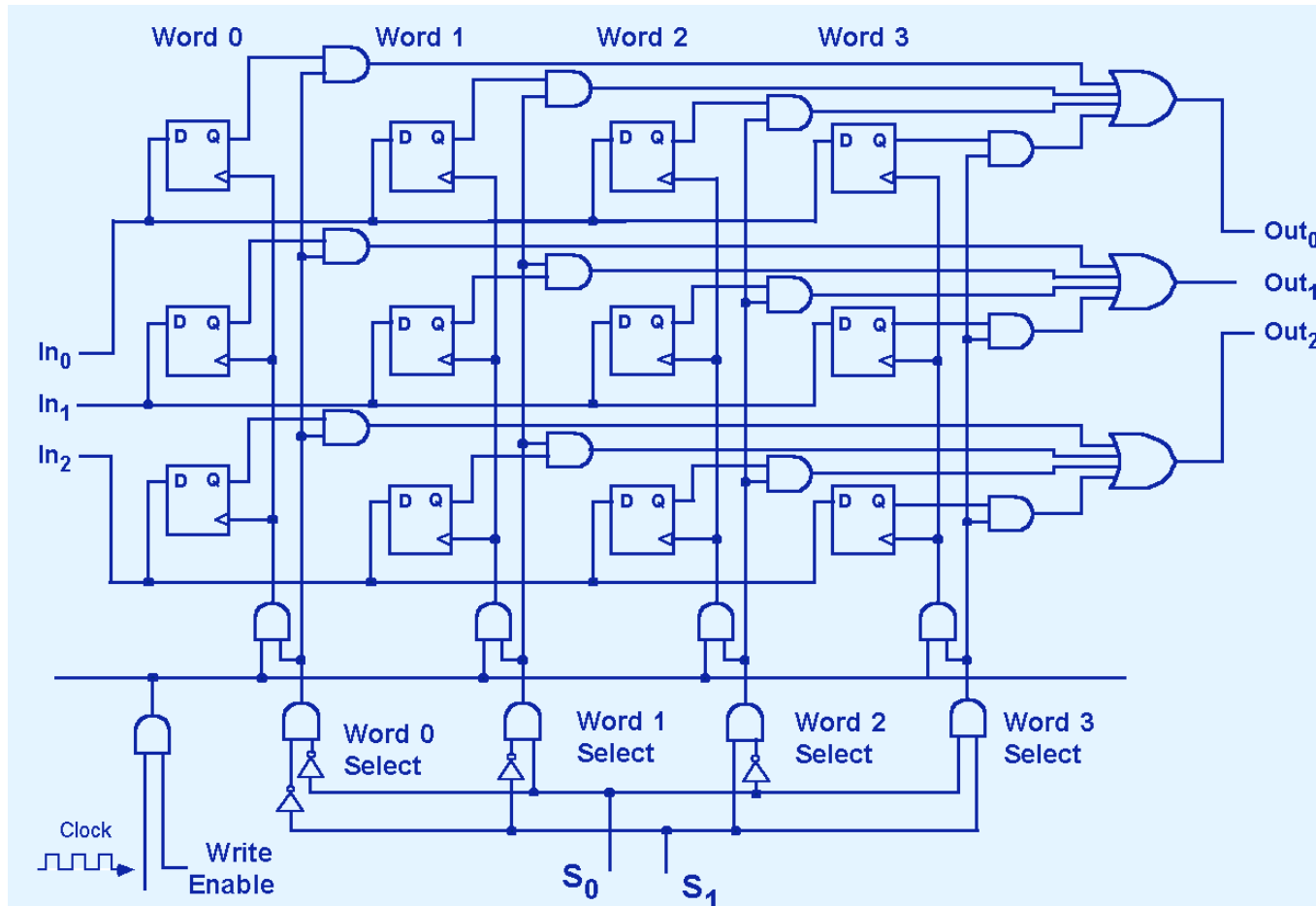
- Sequential circuits are used anytime that we have a “stateful” application.
  - A stateful application is one where the next state of the machine depends on the current state of the machine and the input.
- A stateful application requires both combinational and sequential logic.
- The following slides provide several examples of circuits that fall into this category.

# SEQUENTIAL CIRCUITS: FOUR BIT REGISTER

- This illustration shows a 4-bit register consisting of D flip-flops. You will usually see its block diagram (below) instead.

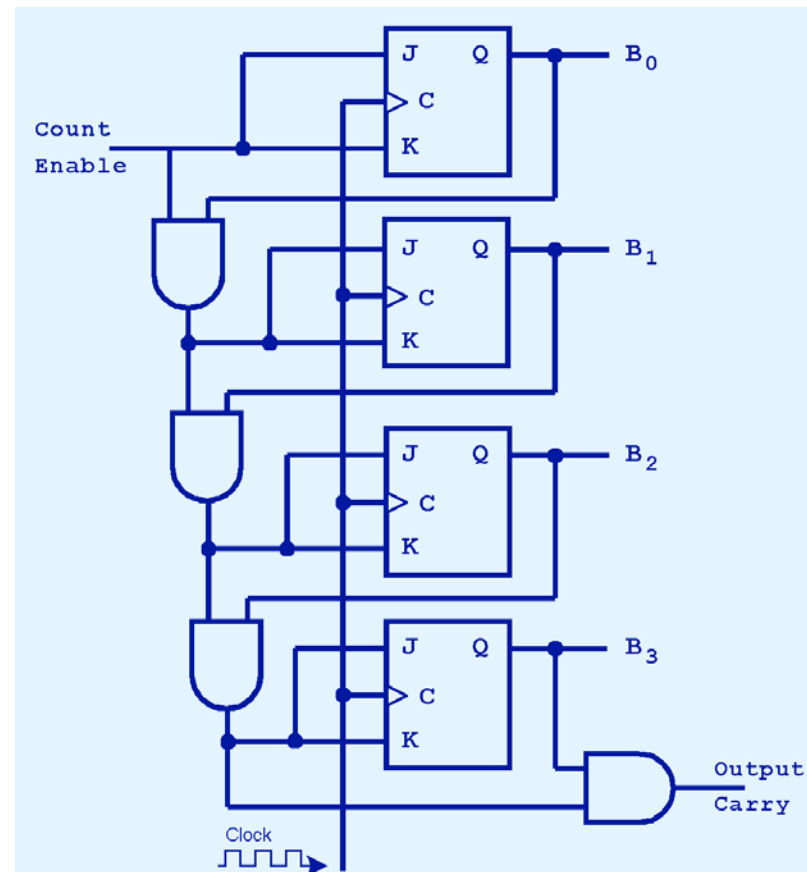


# LARGE MEMORY CONFIGURATION



# BINARY COUNTER

- A binary counter is another example of a sequential circuit.
- The low-order bit is complemented at each clock pulse.
- Whenever it changes from 0 to 1, the next bit is complemented, and so on through the other flip-flops.





## DIGITAL DESIGN: PART ONE

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- We have seen digital circuits from two points of view: digital analysis and digital synthesis.
  - *Digital analysis* explores the relationship between a circuit's inputs and its outputs.
  - *Digital synthesis* creates logic diagrams using the values specified in a truth table.
- Digital systems designers must also be mindful of the physical behaviors of circuits to include minute propagation delays that occur between the time when a circuit's inputs are energized and when the output is accurate and stable.



## DIGITAL DESIGN: PART TWO

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- Digital designers rely on specialized software to create efficient circuits.
  - Thus, software is an enabler for the construction of better hardware.
- Of course, software is in reality a collection of algorithms that could just as well be implemented in hardware.
  - Recall the Principle of Equivalence of Hardware and Software.



## DIGITAL DESIGN: PART THREE

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- When we need to implement a simple, specialized algorithm and its execution speed must be as fast as possible, a hardware solution is often preferred.
- This is the idea behind *embedded systems*, which are small special-purpose computers that we find in many everyday things.
- Embedded systems require special programming that demands an understanding of the operation of digital circuits, the basics of which you have learned in this LESSON.





## BOOLEAN LOGIC: SUMMARY PART ONE

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- Computers are implementations of Boolean logic.
- Boolean functions are completely described by truth tables.
- Logic gates are small circuits that implement Boolean operators.
- The basic gates are AND, OR, and NOT.
  - The XOR gate is very useful in parity checkers and adders.
- The “universal gates” are NOR, and NAND.



## BOOLEAN LOGIC: SUMMARY PART TWO

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- Computer circuits consist of combinational logic circuits and sequential logic circuits.
- Combinational circuits produce outputs (almost) immediately when their inputs change.
- Sequential circuits require clocks to control their changes of state.
- The basic sequential circuit unit is the flip-flop: The behaviors of the SR, JK, and D flip-flops are the most important to know.



## BOOLEAN LOGIC: SUMMARY PART THREE

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- The behavior of sequential circuits can be expressed using characteristic tables or through various finite state machines.
- Moore and Mealy machines are two finite state machines that model high-level circuit behavior.
- Algorithmic state machines are better than Moore and Mealy machines at expressing timing and complex signal interactions.