Mathematical Preliminaries

Developed for the Members of Azera Global

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Sets

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Introduction: Part One

- We have all implicitly dealt with sets
  - Integers ($\mathbb{Z}$), rationals ($\mathbb{Q}$), naturals ($\mathbb{N}$), reals ($\mathbb{R}$), etc.
- We will develop more fully
  - The definitions of sets
  - The properties of sets
  - The operations on sets
- **Definition**: A set is an <u>unordered</u> collection of (<u>unique</u>) objects
- Sets are fundamental discrete structures and for the basis of more complex discrete structures like graphs

Introduction: Part Two

- The objects in a set are called elements or members of a set. A set is said to contain its elements
- Notation, for a set A:
  - $x \in A$: x is an element of A
  - $x \notin A$: x is not an element of A

Properties: Part One

- Two sets, A and B, are equal is they contain the same elements. We write A=B.
- Example:
  - {2,3,5,7}={3,2,7,5}, because a set is unordered
  - Also, {2,3,5,7}={2,2,3,5,3,7} because a set contains unique elements
  - However, {2,3,5,7} $\neq${2,3}

Properties: Part Two

- A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where
  - m_1 occurs a_1 times
  - m_2 occurs a_2 times
  - …
  - m_r occurs a_r times
- In Databases, we distinguish
  - A set: elements cannot be repeated
  - A bag: elements can be repeated

Terminology

- The **set-builder** notation

  $$O=\{ x \mid (x \in \mathbb{Z}) \wedge (x=2k) \text{ for some } k \in Z\}$$

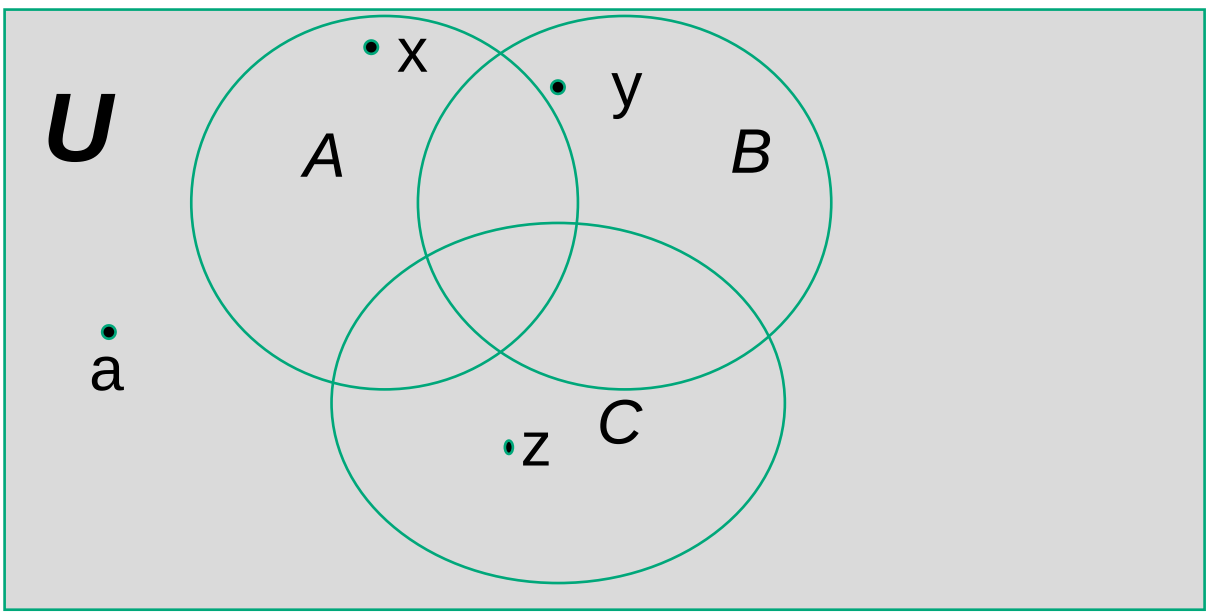
  reads: O is the set that contains all x such that x is an integer and x is even
- A set is defined in **intension** when you give its set-builder notation

  $$O=\{ x \mid (x \in \mathbb{Z}) \wedge (0 \leq x \leq 8) \wedge (x=2k) \text{ for some } k \in Z \}$$
- A set is defined in **extension** when you enumerate all the elements:

  $$O=\{0,2,4,6,8\}$$

Venn Diagram:

- A set can be represented graphically using a Venn Diagram



Properties and Notation: Part One

- A set that has no elements is called the **empty set** or **null set** and is denoted $\varnothing$
- A set that has one element is called a **singleton set**.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}
- Note the subtlety in $\varnothing \neq \{\varnothing\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

Properties and Notation: Part Two

- For any set S
  - $\emptyset \subseteq S$ and
  - $S \subseteq S$
- A is said to be a **subset** of B, and we write $A \subseteq B$, if and only if every element of A is also an element of B
- That is, we have the equivalence:

$$A \subseteq B \Leftrightarrow \forall x\ (x \in A \Rightarrow x \in B)$$

Properties and Notation: Part Three

- A set A that is a subset of a set B is called a **proper subset** if $A \neq B$.
- That is there is an element $x \in B$ such that $x \notin A$
- We write: $A \subset B$,

If there are exactly n distinct elements in a set S, with n a nonnegative integer, we say that:

S is a finite set, and

The cardinality of S is n. Notation: $|S| = n$.

A set that is not finite is said to be infinite

Equivalence: Part One

- To show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$
  - To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B.
- To prove that A is a proper subset of B, you must prove
  - A is a subset of B **and**
  - $\exists x \, (x \in B) \land (x \notin A)$

Equivalence: Part Two

- To show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  - $A \subseteq B$ and
  - $B \subseteq A$
- Logically speaking, you must show the following quantified statements:

$$(\forall x\ (x \in A \Rightarrow x \in B)) \wedge (\forall x\ (x \in B \Rightarrow x \in A))$$

Power Set

- The power set of a set S, denoted P(S), is the set of all subsets of S.
- Examples
  - Let A={a,b,c}, P(A)={∅,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
  - Let A={{a,b},c}, P(A)={∅,{{a,b}},{c},{{a,b},c}}
- Note: the empty set ∅ and the set itself are always elements of the power set.
- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- Let S be a set such that |S|=n, then

$$|P(S)| = 2^n$$

Tuples

- Sometimes we need to consider ordered collections of objects
- The ordered n-tuple $(a_1,a_2,\ldots,a_n)$ is the ordered collection with the element a_i being the i-th element for $i=1,2,\ldots,n$
- A 2-tuple ($n=2$) is called an ordered pair

Cartesian Product

- Let A and B be two sets. The Cartesian product of A and B, denoted AxB, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$AxB = \{ (a,b) \mid (a \in A) \wedge (b \in B) \}$$

- The Cartesian product is also known as the cross product
- A subset of a Cartesian product, $R \subseteq AxB$ is called a relation.
- Note: $AxB \neq BxA$ unless $A=\emptyset$ or $B=\emptyset$ or $A=B$
- Cartesian Products can be generalized for any n-tuple
- The Cartesian product of n sets, $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1,2,\ldots,n \}$$

Notation with Quantifiers

- Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language
- Now we can simplify things using <u>set notation</u>!
- Example
  - $\forall\ x \in \mathbb{R}\ (x^2 \geq 0)$
  - $\exists\ x \in \mathbb{Z}\ (x^2 = 1)$
  - Also mixing quantifiers:

$$\forall a,b,c \in \mathbb{R}\ \exists\ x \in \mathbb{C}\ (ax^2+bx+c=0)$$

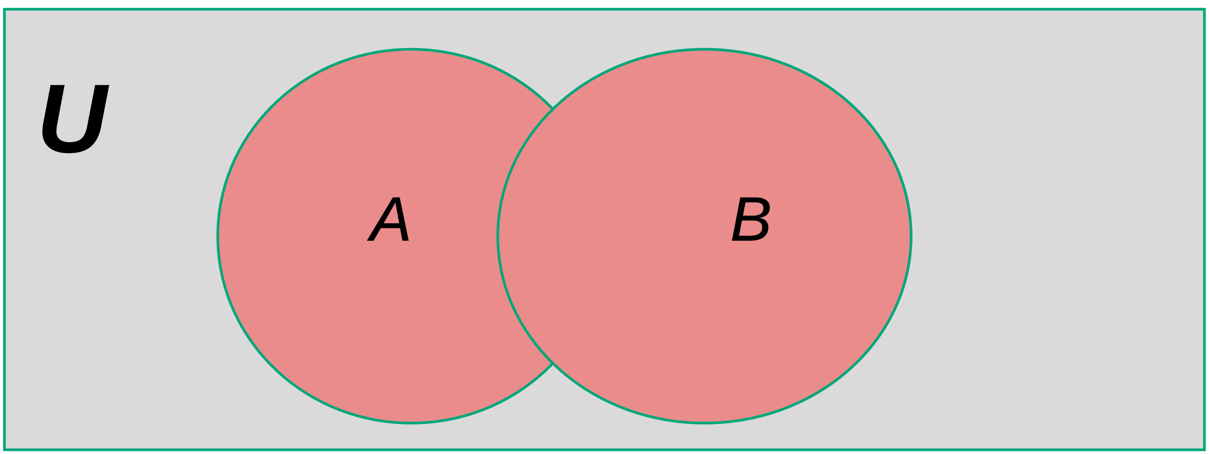
Set Operations

- Arithmetic operators (+,-, $\times$,$\div$) and set operators exist and act on two sets to give us new sets
  - Union
  - Intersection
  - Set difference
  - Set complement
  - Generalized union
  - Generalized intersection

Set Operators: Union

- The union of two sets A and B is the set that contains all elements in A, B, r both. We write:

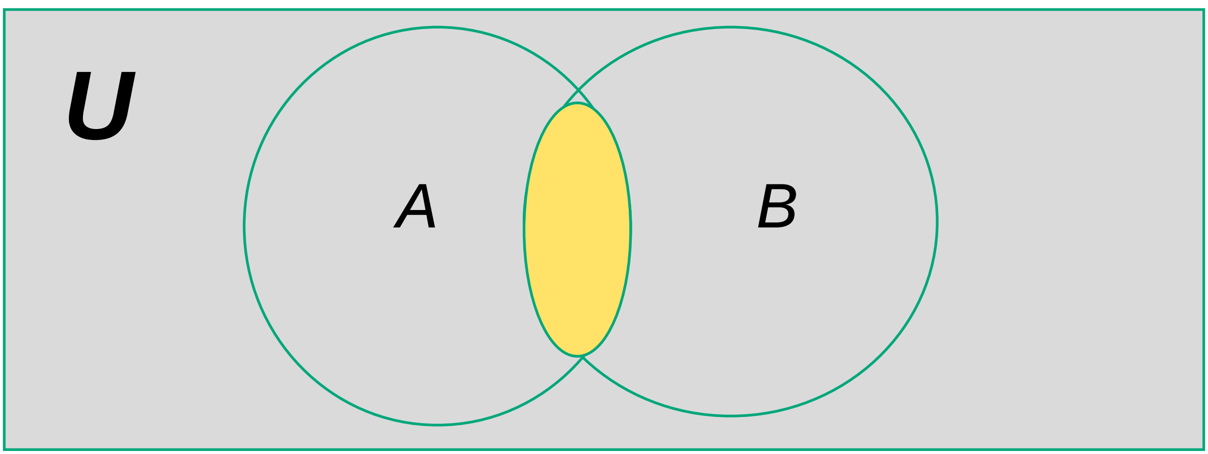
$$A \cup B = \{ x \mid (x \in A) \vee (x \in B) \}$$



Set Operators: Intersection

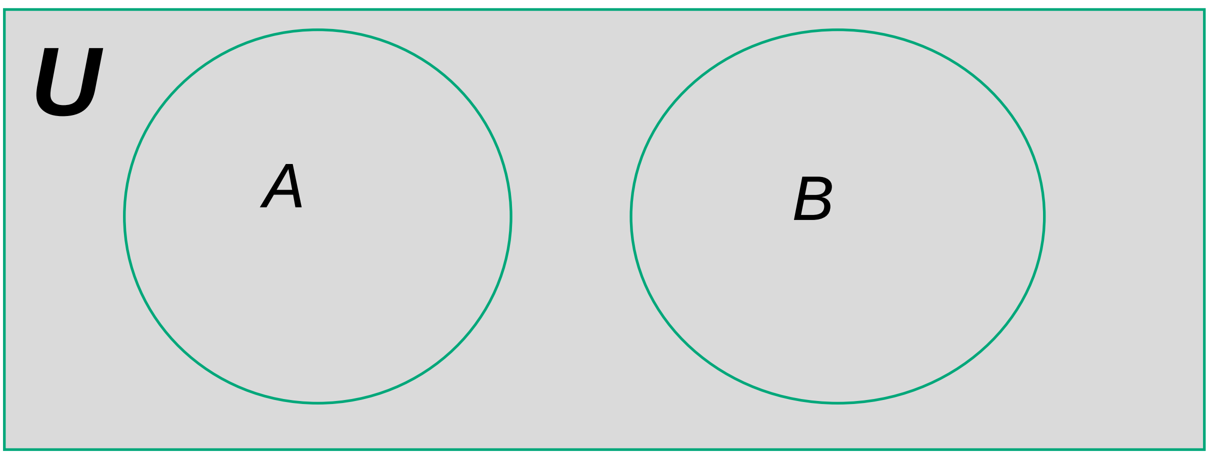
- The intersection of two sets A and B is the set that contains all elements that are element of both A and B. We write:

$$A \cap B = \{ x \mid (x \in A) \wedge (x \in B) \}$$



Disjoint Sets

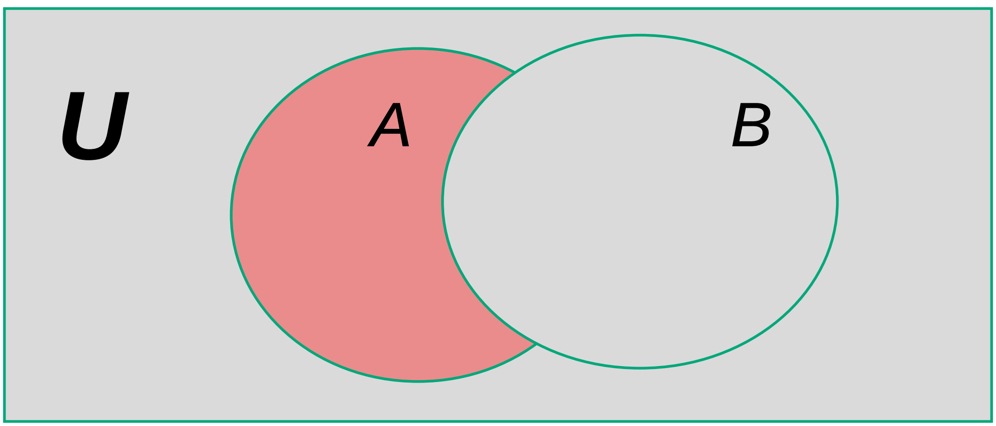
- Two sets are said to be disjoint if their intersection is the empty set: $A \cap B = \varnothing$



Set Difference

- The difference of two sets A and B, denoted A\B or A−B, is the set containing those elements that are in A but not in B

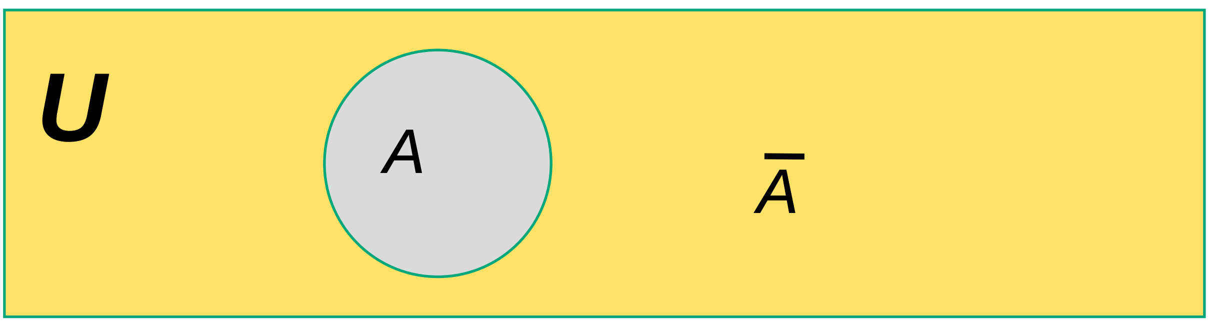
$$A - B = \{ x \mid (x \in A) \wedge (x \notin B) \}$$



Set Complement

- **Definition**: The complement of a set A, denoted $\bar{A}$, consists of all elements <u>not</u> in A. That is the difference of the universal set and U: U\A

$$A = \bar{A} = \{x \mid x \notin A\}$$



Generalized Union

- The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Generalized Intersection

- The intersection of a collection of sets is the set that contains those elements that are members of every set in the collection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Chapter Two: Introduction to Functions

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Relations and Functions

Relation

A **relation** is any set of ordered pairs.

A special kind of relation, called a *function*, is very important in mathematics and its applications.

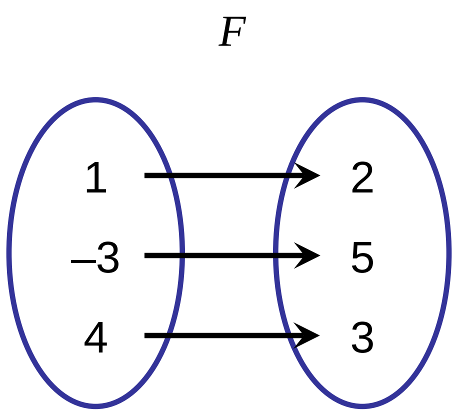
Function

A **function** is a relation in which, for each value of the first component of the ordered pairs, there is *exactly one value* of the second component.

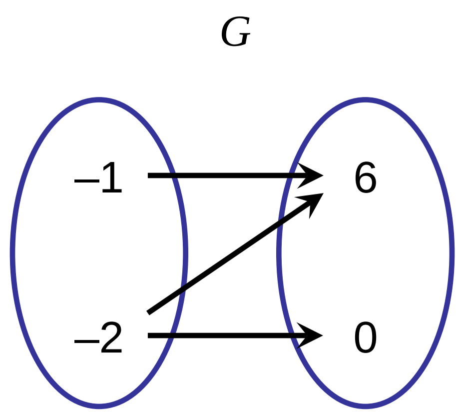
In a relation, the set of all values of the independent variable (x) is the **domain**.

The set of all values of the dependent variable (y) is the **range**

Introduction to Functions



F is a function.

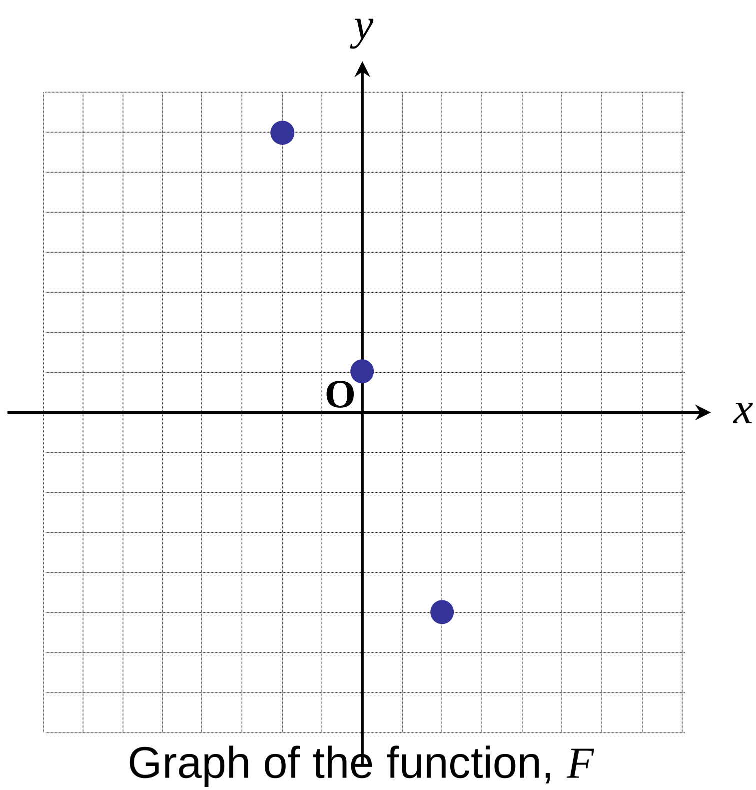


G is not a function.

Tables and Graphs

x	y
–2	6
0	0
2	–6

Table of the function, F



Graph of the function, F

Function Notation

When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say "y is a function of x" to emphasize that y *depends on* x. We use the notation

$$y = f(x),$$

called **function notation**, to express this and read $f(x)$, as "f of x".

The letter f stands for *function*. For example, if $y = 5x - 2$, we can name this function f and write

$$f(x) = 5x - 2.$$

Note that **$f(x)$ *is just another name for the dependent variable* y.**

Linear Function

A function that can be defined by

$$f(x) = ax + b,$$

for real numbers a and b is a **linear function.**

The value *of a* is the slope of *m* of the graph of the function. Before we can draw a graph of our function we must look at the co-ordinate plane or the Cartesian Co-ordinate plane.

The Co-ordinate Plane

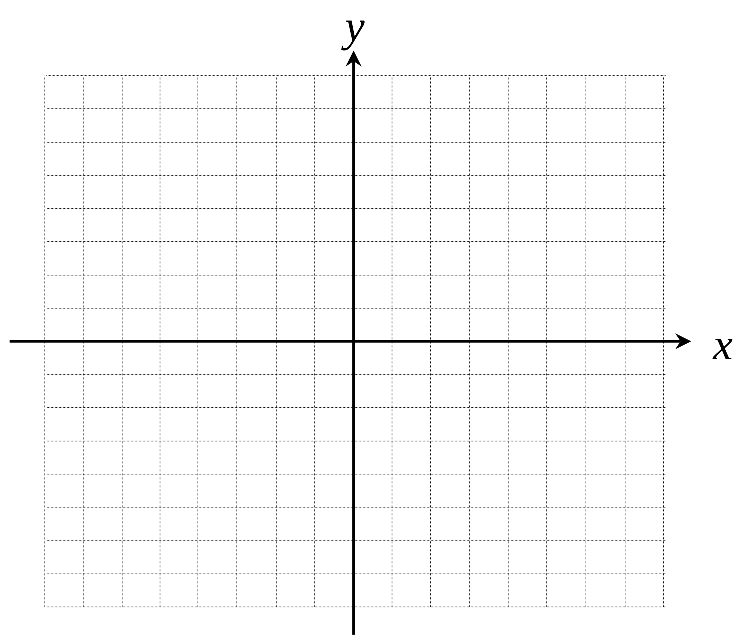
A function that can be defined by $f(x) = ax + b,$

The plane of the grid is called the coordinate plane.

The horizontal number line is called the _x-axis_.

The vertical number line is called the _y-axis_.

The point of intersection of the two axes is called the origin



Graphing a Function

An ordered pair of real numbers, called coordinates of a point, locates a point in the coordinate plane.

Each ordered pair corresponds to EXACTLY <u>one point</u> in the coordinate plane.

The point in the coordinate plane is called the graph of the ordered pair.

Locating a point on the coordinate plane is called <u>graphing</u> the ordered pair.

Chapter Three: Logarithmic Functions

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Definition: Logarithmic Function

For x > 0 and b > 0, b = 1,

$$y = \log_b x \text{ is equivalent to } b^y = x.$$

The function $f(x) = \log_b x$ is the logarithmic function with base b.

Exponent

Logarithmic form: $y = \log_b x$

Base

Exponent

Exponential Form: $b^y = x$.

Base

Properties of Logarithms

For x > 0 and b ≠ 1,

- $\log_b b^x = x$ The logarithm with base b of b raised to a power equals that power.
- $b^{\log_b x} = x$ b raised to the logarithm with base b of a number equals that number.

General Properties: Common Logarithms

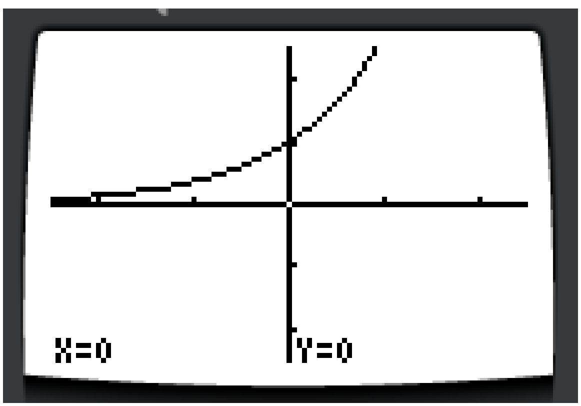
1. $\log_b 1 = 0$	1. $\log 1 = 0$
2. $\log_b b = 1$	2. $\log 10 = 1$
3. $\log_b b^x = 0$	3. $\log 10^x = x$
4. $b^{\log_b x} = x$	4. $10^{\log x} = x$

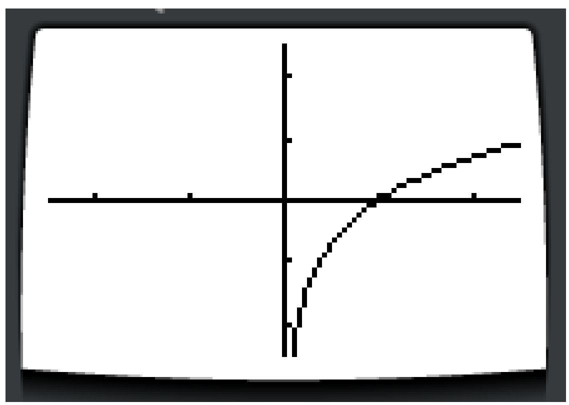
Properties of Natural Logarithms

General Properties	**Natural Logarithms**
1. $\log_b 1 = 0$	1. $\ln 1 = 0$
2. $\log_b b = 1$	2. $\ln e = 1$
3. $\log_b b^x = 0$	3. $\ln e^x = x$
4. $b^{\log_b x} = x$	4. $e^{\ln x} = x$

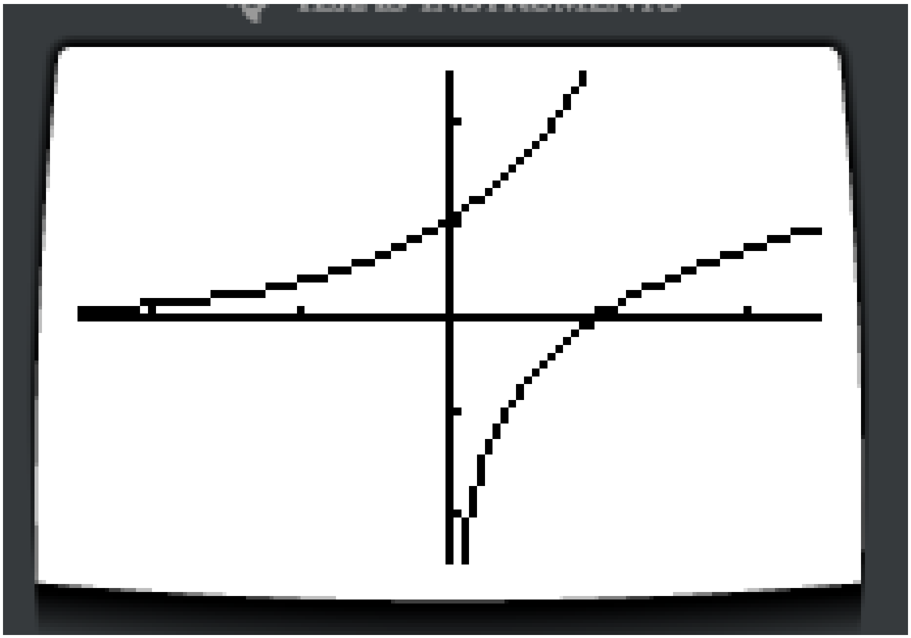
The function $y=e^x$ has an inverse called the Natural Logarithmic Function.

Y=ln x





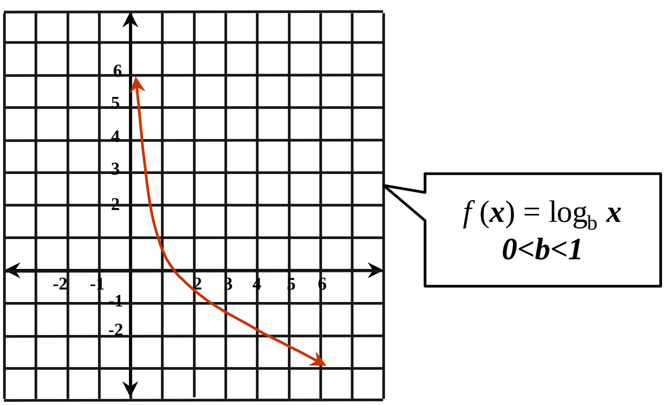
Properties of Natural Logarithms

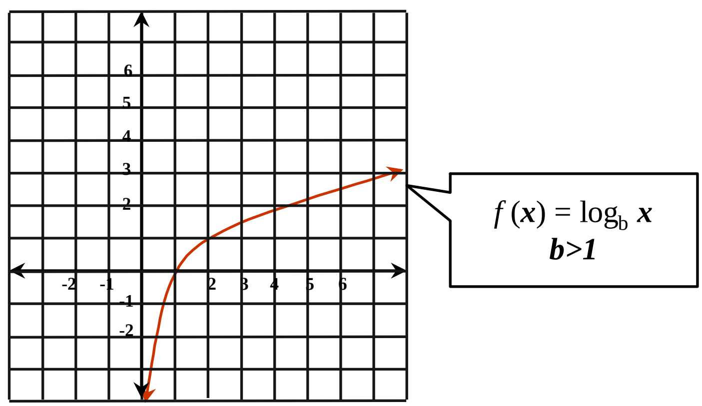


$y=e^x$ and $y=\ln x$ are inverses of each other!

Characteristics of $f(x) = \log_b x$

- The x-intercept is 1. There is no y-intercept.
- The y-axis is a vertical asymptote. $(x = 0)$
- If $0 < b < 1$, the function is decreasing. If $b > 1$, the function is increasing.
- The graph is smooth and continuous. It has no sharp corners or edges.





Domain of Logarithmic Functions

Because the logarithmic function is the inverse of the exponential function, its domain and range are the reversed. $f(x) = \log_b(x + c)$

The domain is { x | $x > 0$ } and the range will be all real numbers.

For variations of the basic graph, say the domain will consist of all x for which $x + c > 0$.

Chapter Four: Trigonometry

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Outline

- Slide 47-Right Triangle Trigonometry
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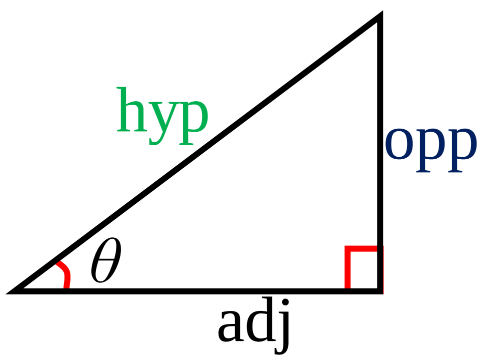
- Slide 49-Trigonometric Ratios
- Slide 50-Reciprocal Functions
- Slide 51-Important Trigonometric Identities

Right Triangle Trigonometry

Trigonometry is based upon ratios of the sides of right triangles.

The six **trigonometric functions** of a right triangle, with an acute angle θ, are defined by **ratios** of two sides of the triangle.



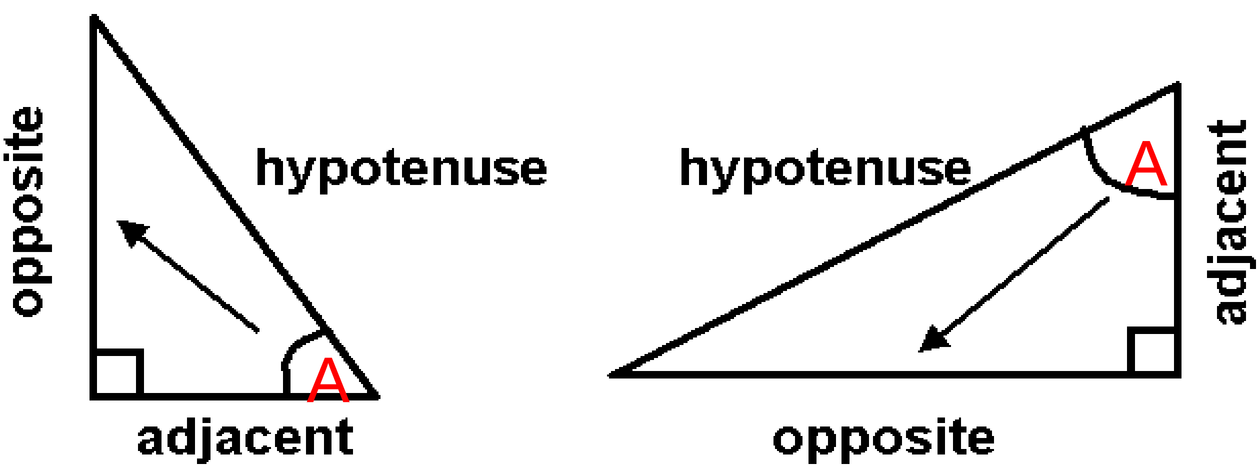
The sides of the right triangle are:

- the side **opposite** the acute angle θ,
- the side **adjacent** to the acute angle θ,
- and the **hypotenuse** of the right triangle.

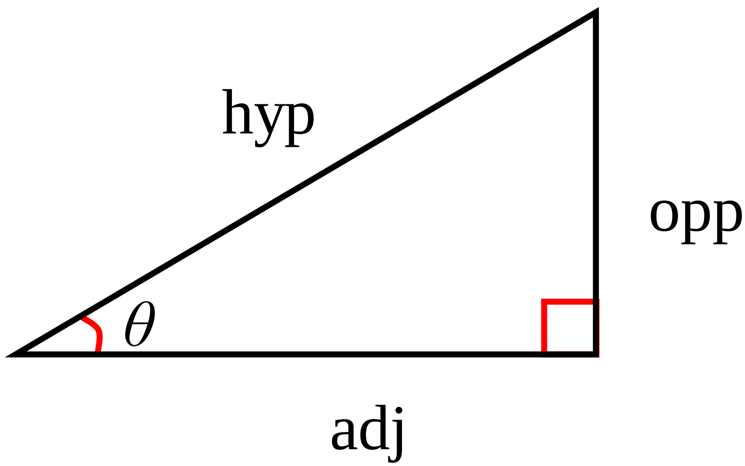
Right Triangle Trigonometry

The hypotenuse is the longest side and is always opposite the right angle.

The opposite and adjacent sides refer to another angle, other than the 90º.



Trigonometric Ratios



The trigonometric functions are:

sine, cosine, tangent, cotangent, secant, and **cosecant**.

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos \theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec \theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

Reciprocal Functions

$\sin\theta = 1/\csc\theta$

$\cos\theta = 1/\sec\theta$

$\tan\theta = 1/\cot\theta$

$\csc\theta = 1/\sin\theta$

$\sec\theta = 1/\cos\theta$

$\cot\theta = 1/\tan\theta$

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Important Trigonometric Identities

Reciprocal Identities

$\sin\theta = 1/\csc\theta$ $\cos\theta = 1/\sec\theta$ $\tan\theta = 1/\cot\theta$

$\cot\theta = 1/\tan\theta$ $\sec\theta = 1/\cos\theta$ $\csc\theta = 1/\sin\theta$

Co function Identities

$\sin\theta = \cos(90^\circ - \theta)$ $\cos\theta = \sin(90^\circ - \theta)$

$\sin\theta = \cos(\pi/2 - \theta)$ $\cos\theta = \sin(\pi/2 - \theta)$

$\tan\theta = \cot(90^\circ - \theta)$ $\cot\theta = \tan(90^\circ - \theta)$

$\tan\theta = \cot(\pi/2 - \theta)$ $\cot\theta = \tan(\pi/2 - \theta)$

$\sec\theta = \csc(90^\circ - \theta)$ $\csc\theta = \sec(90^\circ - \theta)$

$\sec\theta = \csc(\pi/2 - \theta)$ $\csc\theta = \sec(\pi/2 - \theta)$

Quotient Identities

$\tan\theta = \sin\theta/\cos\theta$ $\cot\theta = \cos\theta/\sin\theta$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$

Introduction to Vectors

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Outline

Definition

Vector analysis is a mathematical tool with which electromagnetic (EM) concepts are most conveniently expressed and best comprehended.

A quantity is called a scalar if it has only magnitude (e.g., mass, temperature, electric potential, population).

A quantity is called a vector if it has both magnitude and direction (e.g., velocity, force, electric field intensity).

The magnitude of a vector $\overline{A}$ is a scalar written as A or $|\overline{A}|$

Unit Vector: Part One

A unit vector $\overline{e}_A$ along $|\overline{A}|$ is defined as a vector whose magnitude is unity (that is,1) and its direction is along

$$\overline{e}_A = \frac{\overline{A}}{|\overline{A}|} = \frac{\overline{A}}{A} \qquad (|\overline{e}_A| = 1)$$

Thus: $\overline{A} = A\overline{e}_A$

which completely specifies $\overline{A}$ in terms of A and its direction $\overline{e}_A$

Unit Vector: Part Two

A unit vector $\overline{e}_A$ along $|A|$ is defined as a vector whose magnitude is unity (that is,1) and its direction is along

$$\overline{e}_A = \frac{\overline{A}}{|\overline{A}|} = \frac{\overline{A}}{A} \quad (|\overline{e}_A| = 1) \quad \text{Thus:} \quad \overline{A} = A\overline{e}_A$$

which completely specifies $\overline{A}$ in terms of A and its direction $\overline{e}_A$

A vector $\overline{A}$ in Cartesian (or rectangular) coordinates may be represented as

$$(A_x, A_y, A_z) \quad \text{Where:} \quad A_x\overline{e}_x + A_y\overline{e}_y + A_z\overline{e}_z$$

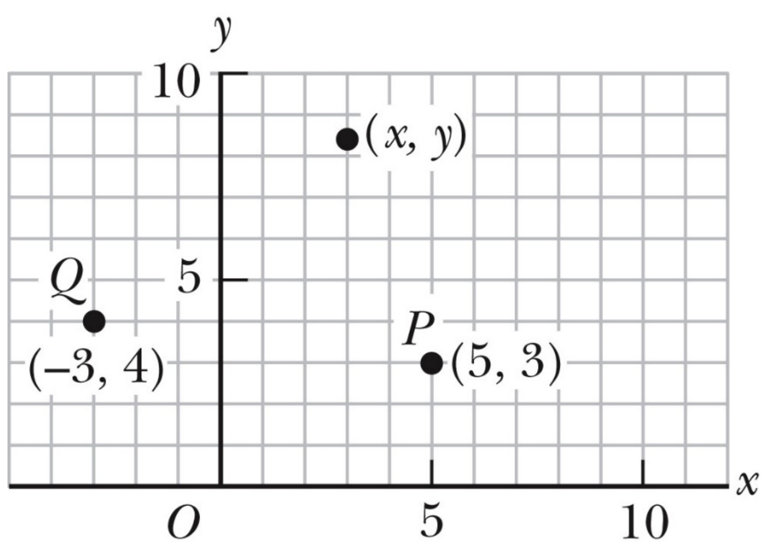
where A_X, A_y, and A_Z are called the components of $\overline{A}$ in the x, y, and z directions, respectively; $\overline{e}_x$, $\overline{e}_y$, and e_z are unit vectors in the x, y and z directions, respectively.

Coordinate Systems

Common coordinate systems are:

- Cartesian
- Polar

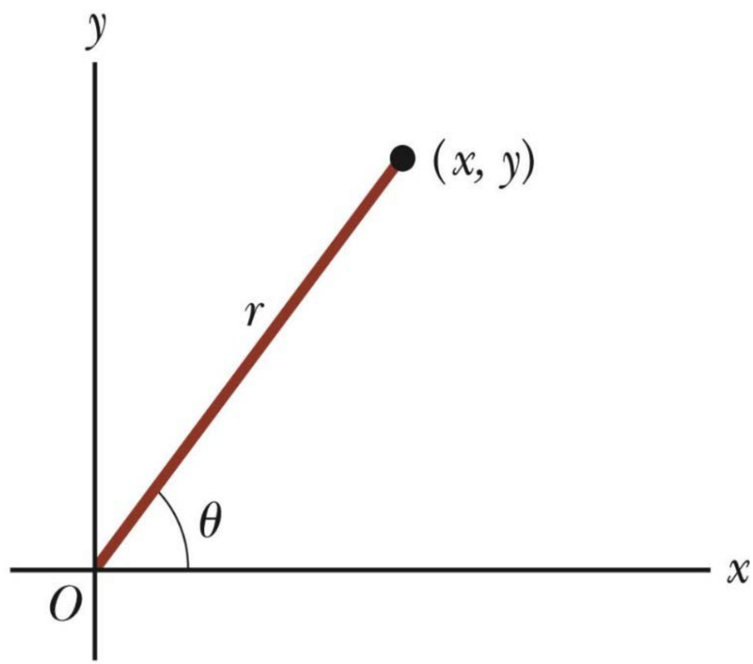
- Also called rectangular coordinate system
- *x*- and *y*- axes intersect at the origin
- Points are labeled (*x*,*y*)



Polar Coordinate System

Origin and reference line are noted

- Point is distance *r* from the origin in the direction of angle θ, ccw from reference line
  - The reference line is often the x-axis.
- Points are labeled (r,θ)



Polar to Cartesian Coordinates

Based on forming a right triangle from r and θ

$$x = r\cos\theta$$

$$y = r\sin\theta$$

If the Cartesian coordinates are known:

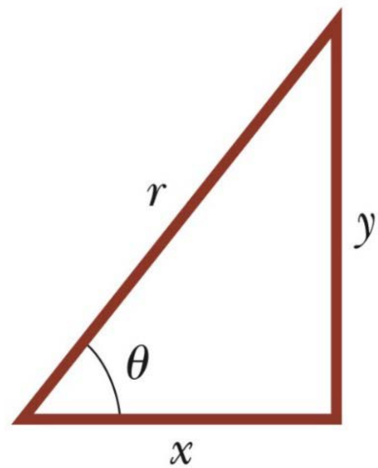
$$\tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



Vector Addition, Rules

The three basic laws of algebra obeyed by any given vector

A, B, and **C,** are summarized as follows:

Commutative $\overline{A} + \overline{B} = \overline{B} + \overline{A}$ $k\overline{A} = \overline{A}k$

Associative $\overline{A} + (\overline{B} + \overline{C}) = (\overline{A} + \overline{B}) + \overline{C}$ $k(l\overline{A}) = (kl)\overline{A}$

Distributive $k(\overline{A} + \overline{B}) = k\overline{A} + k\overline{B}$

where k and l are scalars

Vector Multiplication: Part One

When two vectors $\overline{A}$ and $\overline{B}$ are <u>multiplied</u>, the result is either a scalar or a vector depending on how they are multiplied. The two types of vector multiplication:

1. Scalar (or dot) product: $\overline{A} \cdot \overline{B}$

2. Vector (or cross) product: $\overline{A} \times \overline{B}$

<u>The dot product</u> of the two vectors $\overline{A}$ and $\overline{B}$ is defined geometrically as the product of the magnitude of $\overline{B}$ and The projection of $\overline{A}$ onto $\overline{B}$ (or vice versa):

$$\overline{A} \cdot \overline{B} = AB \cos\theta_{AB}$$

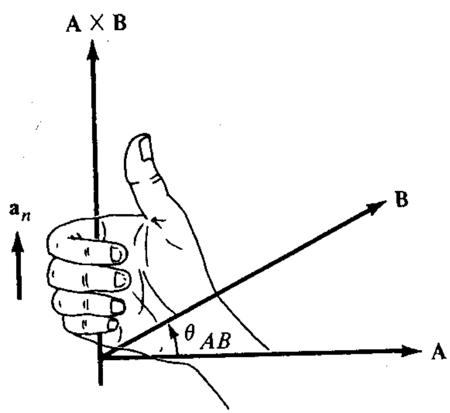
where θ_{AB} is the smaller angle between $\overline{A}$ and $\overline{B}$

Vector Multiplication: Part Two

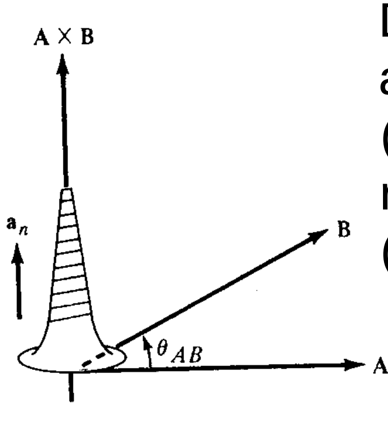
<u>The cross product</u> of two vectors $\overline{A}$ and $\overline{B}$ is defined as

$$\overline{A} \times \overline{B} = AB \sin\theta_{AB} \overline{e}_n$$

where $\overline{e}_n$ is a unit vector normal to the plane containing $\overline{A}$ and $\overline{B}$. The direction of $\overline{e}_n$ is determined using the right-hand rule or the right-handed screw rule.



(a)



(b)

Direction of $\overline{e}_n$ and $\overline{A} \times \overline{B}$ using
(a) right-hand rule,
(b) right-handed screw rule

Vector Multiplication: Part Three

Note that the cross product has the following basic properties:

(i) It is not commutative: $\overline{A} \times \overline{B} \neq \overline{B} \times \overline{A}$

It is anticommutative: $\overline{A} \times \overline{B} = -\overline{B} \times \overline{A}$

(ii) It is not associative: $\overline{A} \times (\overline{B} \times \overline{C}) \neq (\overline{A} \times \overline{B}) \times \overline{C}$

(iii) It is distributive: $\overline{A} \times (\overline{B} + \overline{C}) = \overline{A} \times \overline{B} + \overline{A} \times \overline{C}$

(iv) $\overline{A} \times \overline{A} = 0$ $(\sin\theta = 0)$

Chapter Five
Differential Calculus

Developed for Azera Global

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Differential Calculus

The two basic forms of calculus are

- *differential calculus* and
- *integral calculus*.

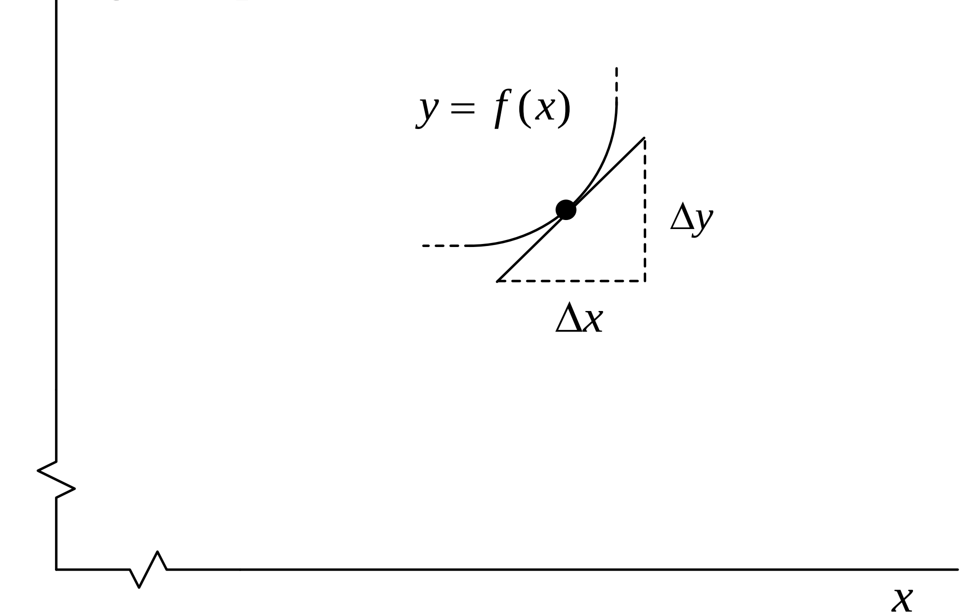
This lecture will be devoted to the former. Integral Calculus will be presented in another lecture.

Differentiation and the Derivative

- The study of calculus begins with the basic definition of a *derivative*. A derivative is obtained through the process of *differentiation,* and the study of all forms of differentiation is collectively referred to as *differential calculus*.
- If we begin with a function and determine its derivative, we arrive at a new function called the *first derivative*.
- If we differentiate the *first derivative*, we arrive at a new function called the *second derivative*, and so on.

Definition of Derivative

The derivative of a function is the *slope* at a given point.



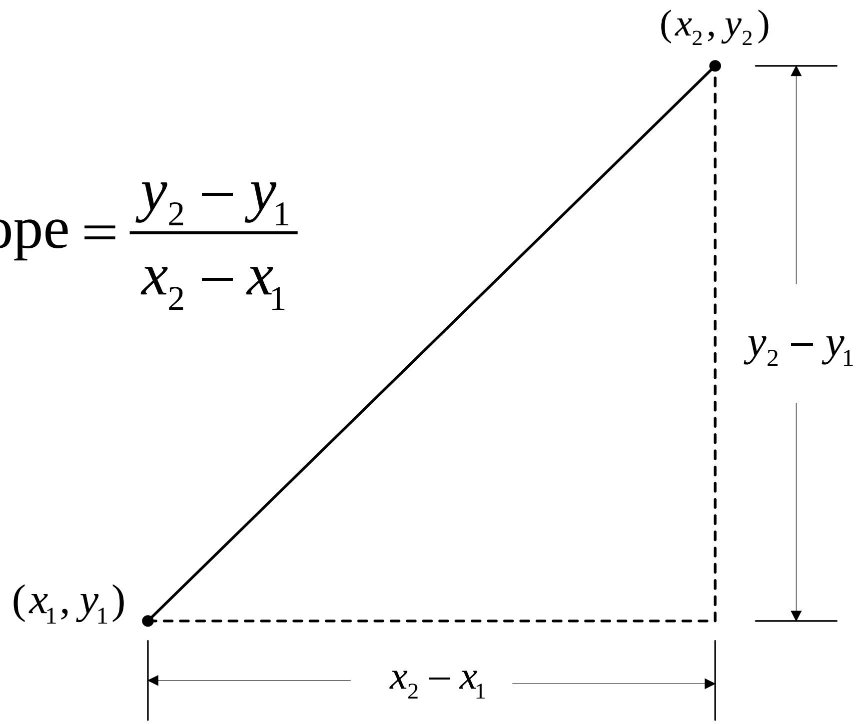
Various Symbols for the Derivative

$$\frac{dy}{dx} \quad \text{or} \quad f\,'(x) \quad \text{or} \quad \frac{df(x)}{dx}$$

Definition: $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Piecewise Linear Segment

$$\frac{dy}{dx} = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$



Example of a Simple Derivative

$$y = x^2$$

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

Chain Rule of Differentiation

$$y = f(u) \qquad u = u(x)$$

$$\frac{dy}{dx} = \frac{df(u)}{du}\frac{du}{dx} = f\,'(u)\frac{du}{dx}$$

where $$f\,'(u) = \frac{df(u)}{du}$$

Table of Derivatives: Part One

$f(x)$	$f'(x)$	Derivative Number
$af(x)$	$af'(x)$	D-1
$u(x)+v(x)$	$u'(x)+v'(x)$	D-2
$f(u)$	$f'(u)\frac{du}{dx}=\frac{df(u)}{du}\frac{du}{dx}$	D-3
a	0	D-4
$x^n \quad (n \neq 0)$	nx^{n-1}	D-5
$u^n \quad (n \neq 0)$	$nu^{n-1}\frac{du}{dx}$	D-6
uv	$u\frac{dv}{dx}+v\frac{du}{dx}$	D-7
$\frac{u}{v}$	$\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$	D-8
e^u	$e^u\frac{du}{dx}$	D-9

Table of Derivatives:Part Two

a^u	$(\ln a)a^u \frac{du}{dx}$		D-10
$\ln u$	$\frac{1}{u}\frac{du}{dx}$		D-11
$\log_a u$	$(\log_a e)\frac{1}{u}\frac{du}{dx}$		D-12
$\sin u$	$\cos u\left(\frac{du}{dx}\right)$		D-13
$\cos u$	$-\sin u\frac{du}{dx}$		D-14
$\tan u$	$\sec^2 u\frac{du}{dx}$		D-15
$\sin^{-1} u$	$\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$	$\left(-\frac{\pi}{2} \le \sin^{-1} u \le \frac{\pi}{2}\right)$	D-16
$\cos^{-1} u$	$\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$	$\left(0 \le \cos^{-1} u \le \pi\right)$	D-17
$\tan^{-1} u$	$\frac{1}{1+u^2}\frac{du}{dx}$	$\left(-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}\right)$	D-18

Higher-Order Derivatives

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x) = \frac{df(x)}{dx}$$

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = f^{(3)}(x) = \frac{d^3 f(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Applications: Maxima and Minima

- 1. Determine the derivative.
- 2. Set the derivative to 0 and solve for values that satisfy the equation.
- 3. Determine the second derivative.
  - (a) If second derivative > 0, point is a *minimum*.
  - (b) If second derivative < 0, point is a *maximum*.

Displacement, Velocity, Acceleration

- Displacement $$y$$
- Velocity $$v = \frac{dy}{dt}$$
- Acceleration $$a = \frac{dv}{dt} = \frac{d^2 y}{dt^2}$$

Partial Derivatives and Gradients

Developed for Azera Global

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Definition: Partial Derivative

- the partial derivative of f(x,y) with respect to x and y are

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \left(\frac{\partial f}{\partial x}\right)_y = f_x$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \left(\frac{\partial f}{\partial y}\right)_x = f_y$$

- second partial derivatives of two-variable function f(x,y)

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

Total Differential

The total differential and total derivative

$$x \to x + \Delta x \text{ and } y \to y + \Delta y \Rightarrow f \to f + \Delta f$$

$$\begin{aligned}\Delta f &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y) \\ &= \left[\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right]\Delta x + \left[\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}\right]\Delta y\end{aligned}$$

as $\Delta x \to 0$ and $\Delta y \to 0$, the total differential df is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

for n-variable function $f(x_1, x_2, \ldots x_n)$

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots\ldots\ldots\ldots + \frac{\partial f}{\partial x_n}dx_n$$

Exact and Inexact Differentials

If a function can be obtained by directly integrating its total differential, the differential of function f is called exact differential, whereas those that do not are inexact differential.

$$\text{(1)}\quad df = xdy + (y+1)dx \Rightarrow f(x,y) = xy + x \qquad \textbf{exact differential}$$

$$\text{(2)}\quad df = xdy + 3ydx$$

$$\Rightarrow \textbf{function}\ \ f(x,y)\ \ \textbf{doesnot exist} \Rightarrow \textbf{inexact differential}$$

Properties of exact differentials:

$$A(x,y)dx + B(x,y)dy = df \Rightarrow \frac{\partial f}{\partial x} = A(x,y) \ \textbf{and}\ \frac{\partial f}{\partial y} = B(x,y)$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial A}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial B}{\partial x} \Rightarrow \frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$$

Properties: Part One

$$x = x(y,z) \Rightarrow dx = (\frac{\partial x}{\partial y})_z dy + (\frac{\partial x}{\partial z})_y dz$$

$$y = y(x,z) \Rightarrow dy = (\frac{\partial y}{\partial x})_z dx + (\frac{\partial y}{\partial z})_x dz$$

$$z = z(x,y) \Rightarrow dz = (\frac{\partial z}{\partial x})_y dx + (\frac{\partial z}{\partial y})_x dy$$

$$dx = (\frac{\partial x}{\partial y})_z (\frac{\partial y}{\partial x})_z dx + [(\frac{\partial x}{\partial y})_z (\frac{\partial y}{\partial z})_x + (\frac{\partial x}{\partial z})_y]dz$$

if z **is a constant** $\Rightarrow dz = 0$

$$(\frac{\partial x}{\partial y})_z = (\frac{\partial y}{\partial x})_z^{-1} \text{ reciprocity relation}$$

if x **is a constant** $\Rightarrow dx = 0$

$$(\frac{\partial y}{\partial z})_x (\frac{\partial z}{\partial x})_y (\frac{\partial x}{\partial y})_z = -1 \text{ cyclic relation}$$

Properties: Part Two

The chain rule

for $f = f(x,y)$ and $x = x(u),\ y = y(u)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow \frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$$

for many variables $f(x_1, x_2, \ldots, x_n)$ and $x_i = x_i(u)$

$$\Rightarrow \frac{df}{du} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{du} = \frac{\partial f}{\partial x_1}\frac{dx_1}{du} + \frac{\partial f}{\partial x_2}\frac{dx_2}{du} + \ldots\ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{du}$$

Partial Differentiation of Integrals

$$F(x,t) = \int f(x,t)dt \Rightarrow \frac{\partial F(x,t)}{\partial x} = f(x,t)$$

$$\Rightarrow \frac{\partial^2 F(x,t)}{\partial t \partial x} = \frac{\partial^2 F(x,t)}{\partial x \partial t} \Rightarrow \frac{\partial}{\partial t}[\frac{\partial F(x,t)}{\partial x}] = \frac{\partial}{\partial x}[\frac{\partial F(x,t)}{\partial t}] = \frac{\partial f(x,t)}{\partial x}$$

$$\Rightarrow \int \frac{\partial}{\partial t}[\frac{\partial F(x,t)}{\partial x}]dt = \int \frac{\partial}{\partial x} f(x,t)dt \Rightarrow \frac{\partial F(x,t)}{\partial x} = \int \frac{\partial f(x,t)}{\partial x}dt$$

Directional Derivatives: Part One

- Recall that, if z = f(x, y), then the partial derivatives f_x and f_y are defined as:

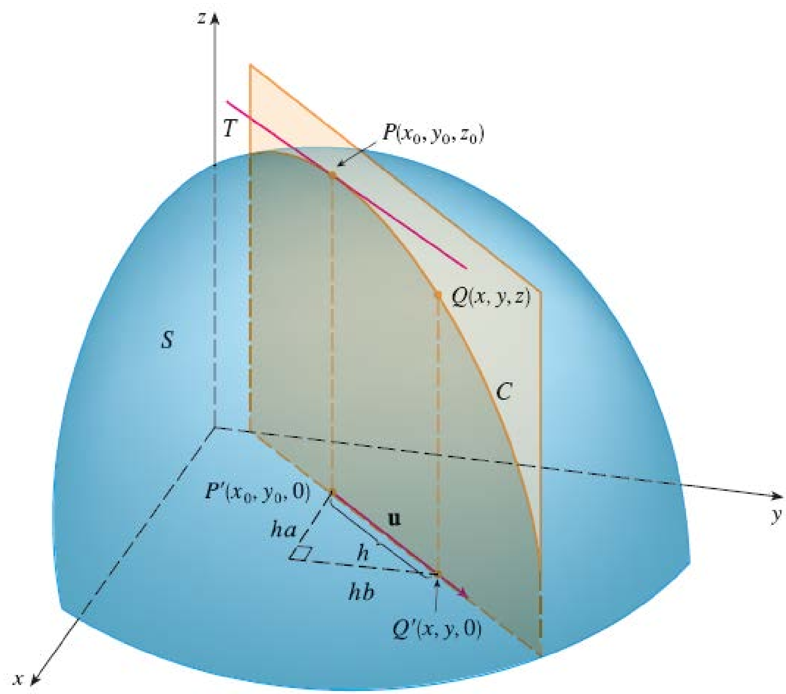
$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Directional Derivatives: Part Two

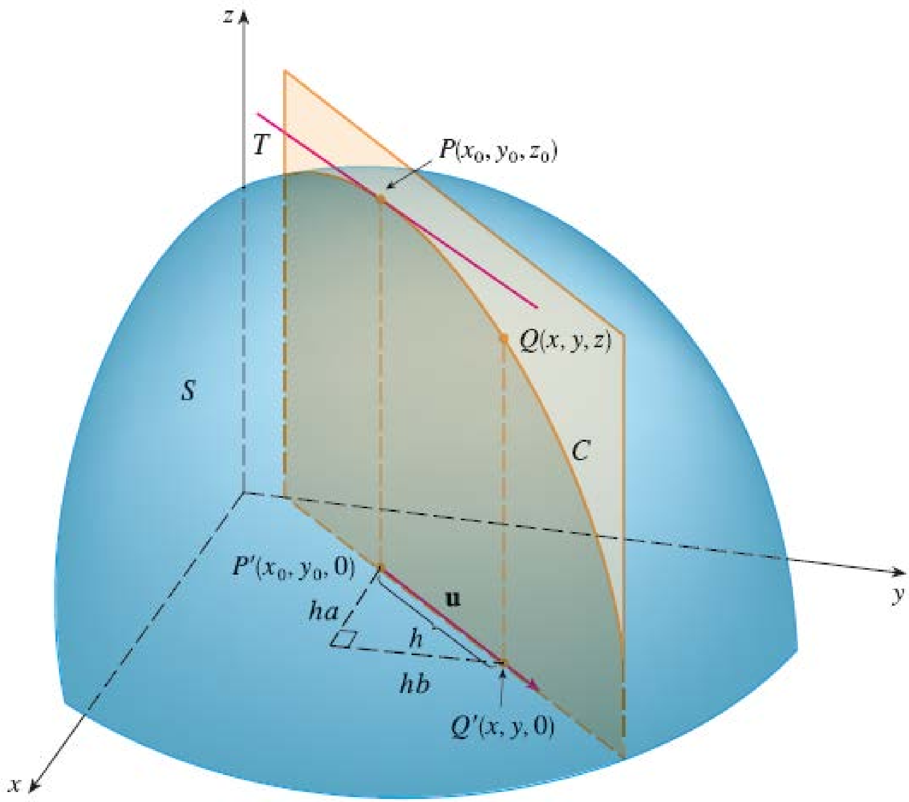
- Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$.

- To do this, we consider the surface S with equation $z = f(x, y)$ [the graph of f] and we let $z_0 = f(x_0, y_0)$.
- Then, the point $P(x_0, y_0, z_0)$ lies on S.



Directional Derivatives: Part Three

- The vertical plane that passes through *P* in the direction of **u** intersects *S* in a curve *C*.
- The slope of the tangent line *T* to *C* at the point *P* is the rate of change of *z* in the direction of **u**.



Directional Derivatives: Part Four

Now, let:

$Q(x, y, z)$ be another point on C.

P', Q' be the projections of P, Q on the xy-plane.

Then the vector $\overrightarrow{P'Q'}$ is parallel to **U**.

So: $\overrightarrow{P'Q'} = h\mathbf{u}$

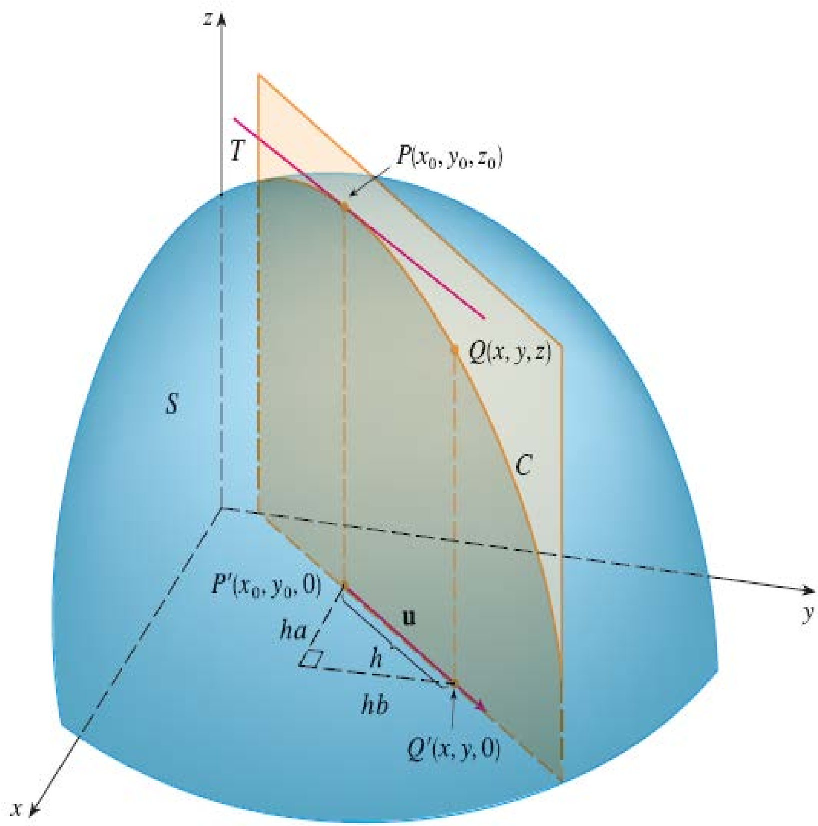
$$= \langle ha, hb \rangle$$

For some scaler h.

Therefore:

$$x - x_0 = ha$$

$$y - y_0 = hb$$



Directional Derivatives: Part Five

From: $x - x_0 = ha$

$y - y_0 = hb$

Then:

$$\frac{\Delta z}{h} = \frac{z - z_0}{h}$$

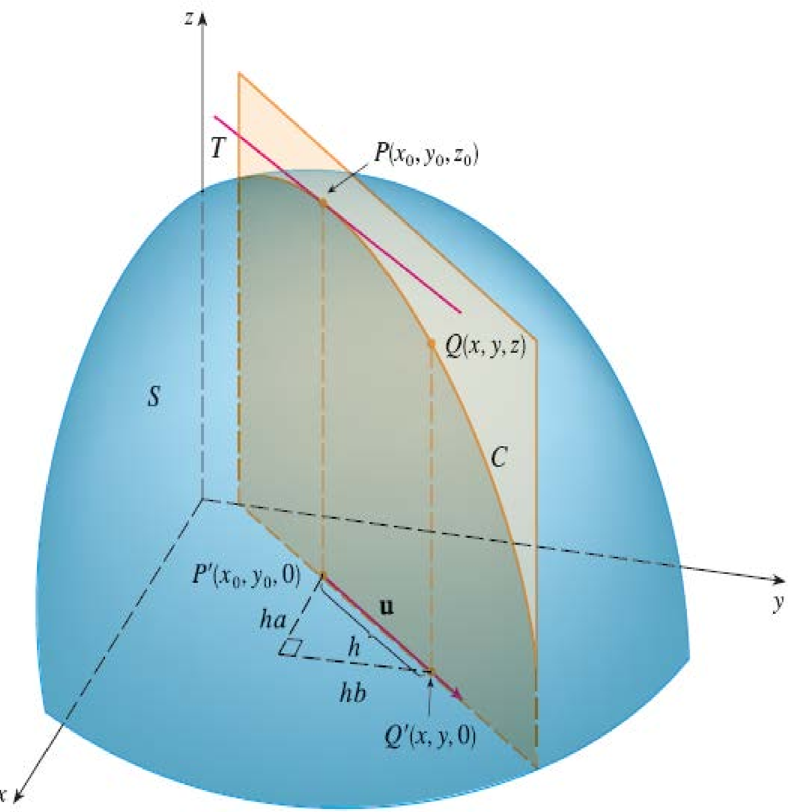
$$= \frac{f(x_0 + ha, y_0 + hb) - f(x_{0,} y_0)}{h}$$

In the limit as $h \to 0$, we obtain the rate of change of z in the direction of **U.**

This is called the directional derivative of f in the direction of **U.**

$$D_{\mathbf{u}} f(x_0, y_0)$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$



Directional Derivatives: Part Six

If we define a function *g* of the single variable *h* by

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

If we define a function *g* of the single variable *h* by:

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have the following equation.

$$g'(0)$$

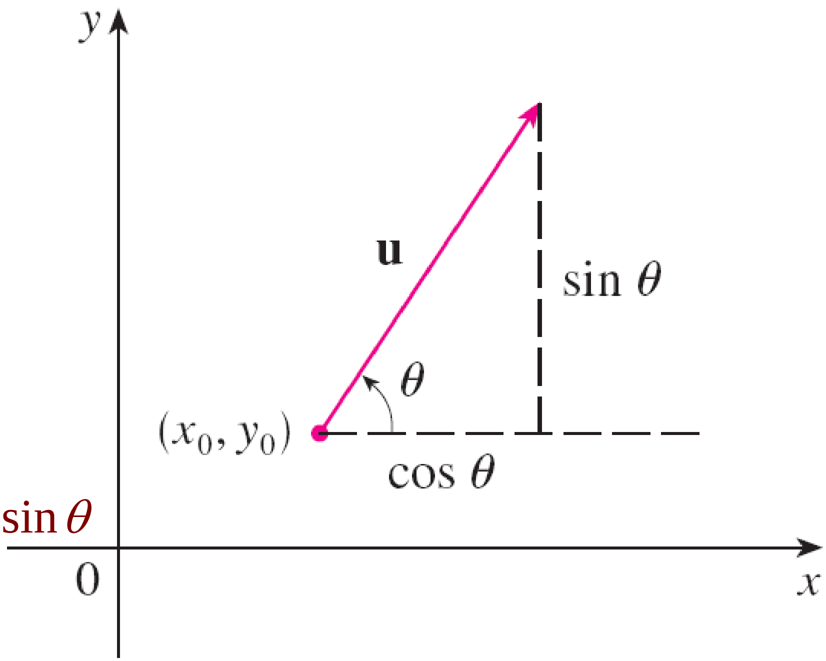
$$= \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_{\mathbf{u}} f(x_0, y_0)$$

Directional Derivatives: Part Seven

Suppose the unit vector **u** makes an angle θ with the positive x-axis, as shown. Then, we can write **u** = $\langle \cos\theta, \sin\theta \rangle$ and the directional derivative becomes:



$$D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

Notice that the directional derivative can be written as the dot product of two vectors:

$$\begin{aligned} D_{\mathbf{u}} f(x,y) &= f_x(x,y)\,a + f_y(x,y)\,b \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u} \end{aligned}$$

The Gradient: Part One

The first vector in that dot product occurs not only in computing directional derivatives but in many other contexts as well. This directional derivative is called the Gradient of *f*.

The Gradient of *f* is written as: ∇f which is read as “del *f*”

If *f* is a function of two variables *x* and *y* then the gradient of *f(x,y)* is defined as:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial x}\mathbf{j}$$

We can rewrite the expression for the directional derivative as:

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

The Gradient: Part Two

For functions of three variables, we can define directional derivatives in a similar manner.

The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector **u** = $\langle a, b, c \rangle$ is:

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

Using vector notation we can rewrite the directional derivative as:

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where:

- $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$
- $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$

The Gradient: Part Three

For a function *f* of three variables, the gradient vector, denoted by ∇f or grad *f*, is:

$$\nabla f(x, y, z)$$
$$= \langle f_x(x, y, z), f_y(x, y, z,), f_z(x, y, z) \rangle$$

And is written as: $\nabla f = \langle f_x, f_y, f_z \rangle$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

The directional derivative can be rewritten as:

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is: $|\nabla f(\mathbf{x})|$ and it occurs when **u** has the same direction as the gradient vector $\nabla f(\mathbf{x})$

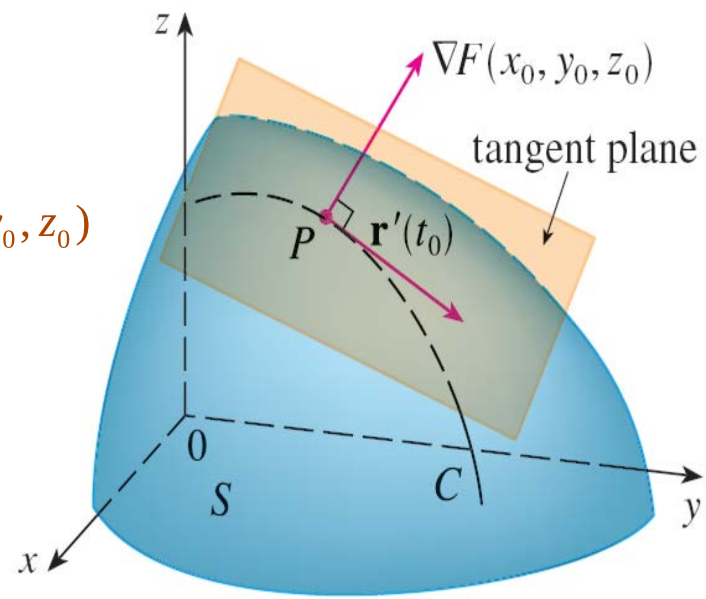
Tangent Plane

Suppose *S* is a surface with equation *F*(*x*, *y*, *z*) that is, it is a level surface of a function *F* of three variables.

Then, let $P(x_0, y_0, z_0)$ be a point on *S*.

Then, let *C* be any curve that lies on the surface *S* and passes through the point *P*.

The curve *C* is described by a continuous vector function
$\mathbf{r}(t) = <x(t), y(t), z(t)>$
The gradient vector at *P* $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ and to any curve *C* on *S* that passes through *P*. Thus the direction of the normal line is given by the gradient vector. $\nabla F(x_0, y_0, z_0)$

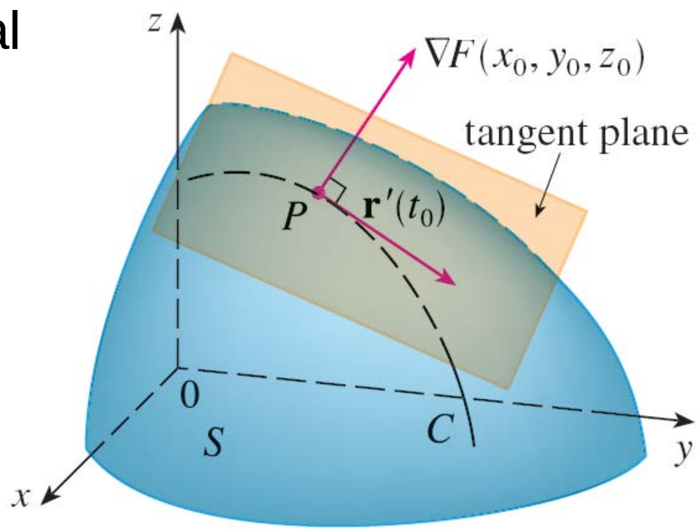


Summary of Gradient

We now summarize the ways in which the gradient vector is significant.

For a function *f* of three variables and a point $P(x_0, y_0, z_0)$ in its domain we know that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of *f*.

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface *S* of *f* through *P*.
So, it seems reasonable that, if we move in the perpendicular direction, we get the maximum increase.



Chapter Six:
Integral Calculus

Developed for Azera Global

By: Joseph D. Fournier B.Sc.E.E., M.Sc.E.E.

Integral Calculus

- The basic concepts of *differential calculus* were covered in the preceding presentation. This presentation will be devoted to *integral calculus*, which is the other broad area of calculus.

Anti-Derivatives

An anti-derivative of a function $f(x)$ is a new function $F(x)$ such that

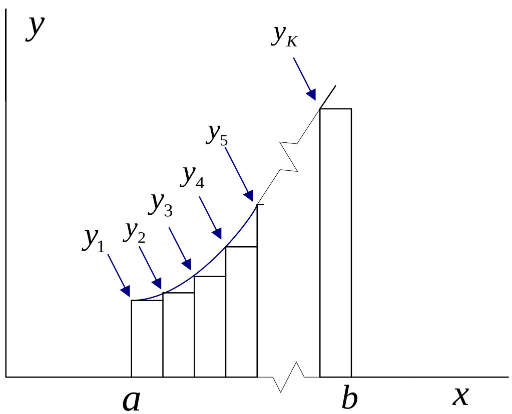
$$\frac{dF(x)}{dx} = f(x)$$

Indefinite and Definite Integrals

Indefinite $$\int f(x)dx$$

Definite $$\int_{x_1}^{x_2} f(x)dx$$

Definite Integral/ Area Under the Curve



$$\text{Approximate Area} = \sum_k y_k \Delta x$$

Exact Area as Definite Integral

$$\int_a^b y\,dx = \lim_{\Delta x \to dx} \sum_k y_k \Delta x$$

Definite Integral with Variable Upper Limit

$$\int_a^x y\,dx$$

More "proper" form with "dummy" variable

$$\int_a^x y(u)\,du$$

Guidelines

- If y is a non-zero constant, integral is either increasing or decreasing linearly.
- If segment is triangular, integral is increasing or decreasing as a parabola.
- If $y=0$, integral remains at previous level.
- Integral moves up or down from previous level; i.e., no sudden jumps.
- Beginning and end points are good reference levels.

Tabulation of Integrals

$$F(x) = \int f(x)dx$$

$$I = \int_a^b f(x)dx$$

$$I = F(x)\Big]_a^b = F(b) - F(a)$$

Common Integrals: Part One

$f(x)$	$F(x)=\int f(x)dx$	Integral Number
$af(x)$	$aF(x)$	I-1
$u(x)+v(x)$	$\int u(x)dx+\int v(x)dx$	I-2
a	ax	I-3
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	I-4
e^{ax}	$\frac{e^{ax}}{a}$	I-5
$\frac{1}{x}$	$\ln x$	I-6
$\sin ax$	$-\frac{1}{a}\cos ax$	I-7
$\cos ax$	$\frac{1}{a}\sin ax$	I-8
$\sin^2 ax$	$\frac{1}{2}x-\frac{1}{4a}\sin 2ax$	I-9

Common Integrals: Part Two

$\cos^2 ax$	$\frac{1}{2}x + \frac{1}{4a}\sin 2ax$	I-10
$x\sin ax$	$\frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$	I-11
$x\cos ax$	$\frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$	I-12
$\sin ax\cos ax$	$\frac{1}{2a}\sin^2 ax$	I-13
$\sin ax\cos bx$ for $a^2 \neq b^2$	$-\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$	I-14
xe^{ax}	$\frac{e^{ax}}{a^2}(ax-1)$	I-15
$\ln x$	$x(\ln x - 1)$	I-16
$\frac{1}{ax^2+b}$	$\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{a}{b}}\right)$	I-17

Displacement, Velocity, Acceleration

$$a = a(t) = \text{acceleration in meters/second}^2 \ (\text{m/s}^2)$$

$$v = v(t) = \text{velocity in meters/second (m/s)}$$

$$y = y(t) = \text{displacement in meters } (\text{m})$$

$$\frac{dv}{dt} = a(t) \quad dv = \left(\frac{dv}{dt}\right)dt = a(t)dt \quad \int dv = \int a(t)dt \quad v = \int a(t)dt + C_1$$

$$\int dv = v \qquad dy = \left(\frac{dy}{dt}\right)dt = v(t)dt$$

$$\frac{dy}{dt} = v(t) \qquad y = \int v(t)dt + C_2$$

Chapter Seven: Complex Variables

Developed for the Azera Group

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Complex Algebra: Part One

Functions of a complex variable provide some powerful and widely useful tools in Engineering and physics.

- Some important physical quantities are complex variables (the wave-function Ψ)
- Evaluating definite integrals.
- Obtaining asymptotic solutions of differentials equations.
- Integral transforms
- Many Physical quantities that were originally real become complex as simple theory is made more general. The energy $E_n \to E_n^0 + i\Gamma$ ($1/\Gamma \to$ the finite life time).

A complex number z = (x,y) = x + iy, Where. $i = \sqrt{-1}$

Complex numbers first arose from the solution of quadratic equations of the type:

$$x^2 + 1 = 0$$

Complex Algebra: Part Two

Although both parts of the complex number are real the ordering of two real numbers (x,y) is significant,

- X: the real part, labeled by **Re(z);**
- y: the imaginary part, labeled by **Im(z)**

The two representations:

(1) Cartesian: x+iy

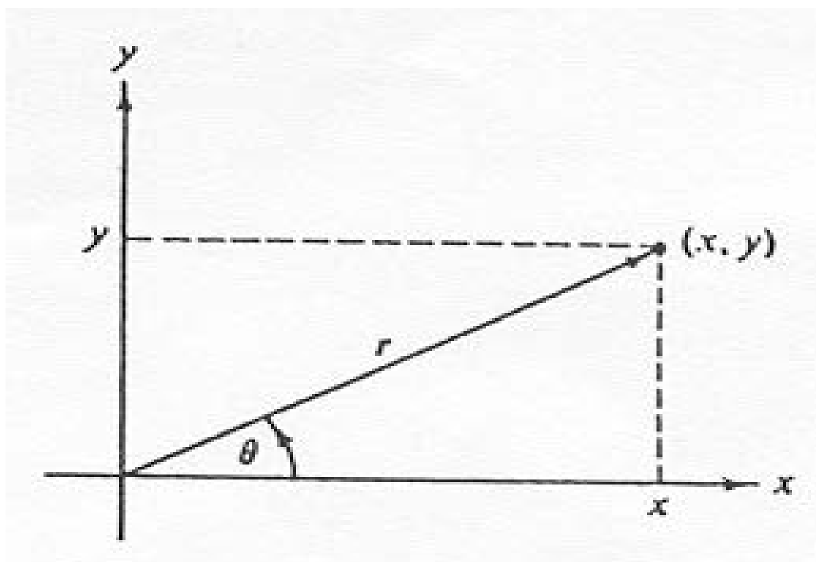
(2) polar representation:

z=r(cos θ + i sinθ) or

$$z = r \cdot e^{i\theta}$$

r – the modulus of z

θ- the argument of z



Complex Algebra: Part Three

The relation between Cartesian and polar representation:

$$r = |z| = \left(x^2 + y^2\right)^{1/2}$$

$$\theta = \tan^{-1}(y/x)$$

The choice of polar representation or Cartesian representation is a matter of convenience. Addition and subtraction of complex variables are easier in the Cartesian representation.

Multiplication, division, powers, roots are easier to handle in polar form,

Complex Algebra: Part Four

Using Cartesian Co-ordinates:

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_2)$$

Using polar co-ordinates:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$z_1 / z_2 = (r_1 / r_2) e^{i(\theta_1 - \theta_2)}$$

$$z^n = r^n e^{in\theta}$$

Complex Algebra: Part Five

Using the polar form,

$$|z_1 z_2| = |z_1||z_2|$$

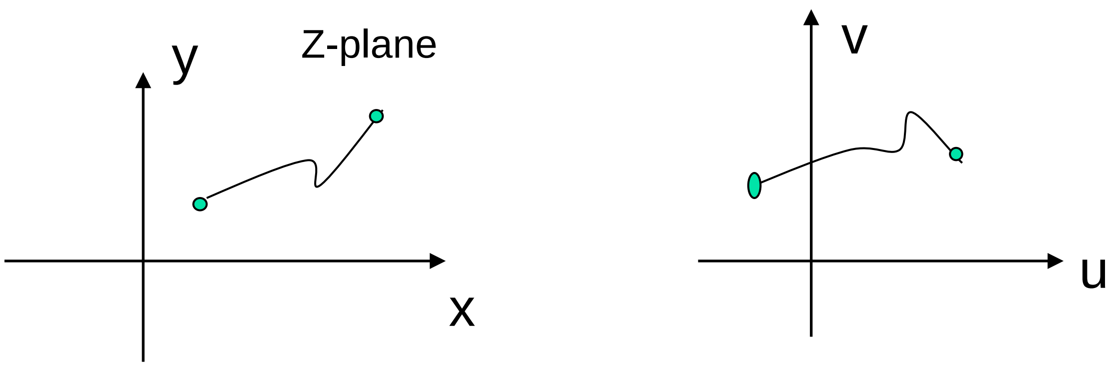
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

From z, complex functions f(z) may be constructed. They can be written f(z) = u(x,y) + iv(x,y) in which v and u are real functions.

For example if $f(z) = z^2$, we have $f(z) = (x^2 - y^2) + i2xy$

The relationship between z and f(z) is best pictured as a mapping operation, we address it in detail later.

Function: Mapping operation



The function *w*(x,y)=*u*(x,y)+i*v*(x,y) maps points in the *xy* plane into points in the *uv* plane.

Since

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n$$

We get a not so obvious formula

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

Complex Conjugation

Replacing i by –i, which is denoted by (*),

$$z^* = x - iy$$

We then have

$$zz^* = x^2 + y^2 = r^2 \qquad |z| = (zz^*)^{1/2}$$

Note: $z = re^{i\theta}$ $\qquad re^{i(\theta+2n\pi)}$

ln z is a multi-valued function. We usually set n=0 and limit the phase to an interval of length of 2π. The value of lnz with n=0 is called the principal value of lnz.

$$\ln z = \ln r + i\theta \qquad \ln z = \ln r + i(\theta + 2n\pi)$$

Another possibility

$|\sin x|, |\cos x| \leq 1$ for a real x;

however, possibly $|\sin z|, |\cos z| > 1$ and even $\rightarrow \infty$

Using the identities:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}; \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

to show (a) $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$

(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Cauchy – Riemann: Part One

Having established complex functions, we now proceed to differentiate them. The derivative of f(z), like that of a real function, is defined by

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z)$$

provided that the limit is independent of the particular approach to the point z. For real variable, we require that

,

$$\lim_{x \to x_o^+} f'(x) = \lim_{x \to x_o^-} f'(x) = f'(x_o)$$

Now, with z (or z_0) some point in a plane, our requirement that the limit be independent of the direction of approach is very restrictive.

Cauchy–Riemann: Part Two

Consider

$$\delta z = \delta x + i\,\delta y$$

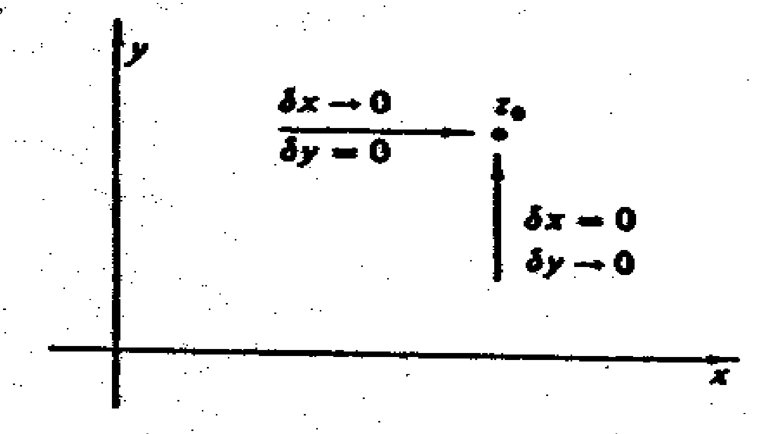
$$\delta f = \delta u + i\,\delta v$$

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

,

Let us take limit by the two different approaches as in the figure. First,

with δy = 0, we let δx→0,



$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\,\frac{\delta v}{\delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

Cauchy–Riemann: Part Three

Assuming the partial derivatives exist. For a second approach, we set $\delta x = 0$ and then let $\delta y \rightarrow 0$. This leads to

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

If we have a derivative, the above two results must be identical. So,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are the famous Cauchy-Riemann conditions. These Cauchy-Riemann conditions are necessary for the existence of a derivative, that is, if exists, the C-R conditions must hold.

Conversely, if the C-R conditions are satisfied and the partial derivatives of u(x,y) and v(x,y) are continuous, then exists.

Analytic functions: Part One

If f(z) is differentiable at $z = z_0$ and in some small region around $z = z_0$,we say that f(z) is analytic at z_0

- Differentiable: If Cauchy-Riemann conditions are satisfied the partial derivatives of u and v are continuous

For Analytic functions: $\nabla^2 u = \nabla^2 v = 0$

- For integration: In close analogy to the integral of a real function, The contour $z_0 \to z_0'$ is divided into n $n \to \infty$ intervals .Let $|\Delta z_j| = |z_j - z_{j-1}| \to 0$ for j. Then

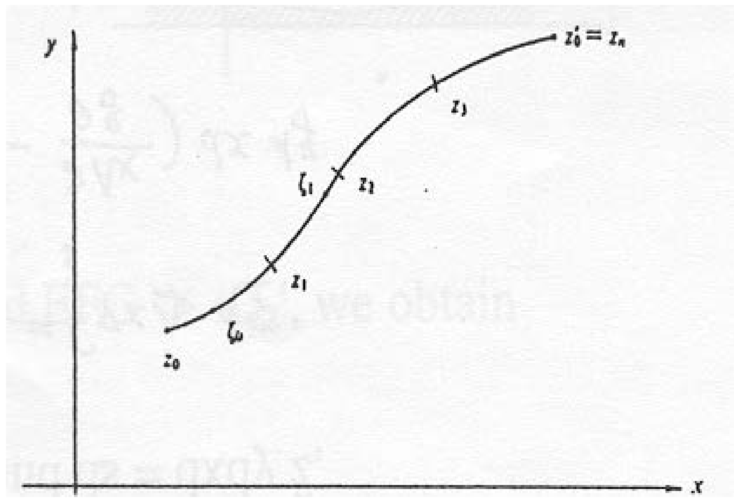
$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)\Delta z_j = \int_{z_0}^{z_0'} f(z)dz$$

The right-hand side of the above equation is called the contour (path) integral of f(z)

Analytic functions: Part Two

$$\lim_{n\to\infty}\sum_{j=1}^{n} f(\zeta_j)\Delta z_j = \int_{z_0}^{z_0'} f(z)\,dz$$

provided that the limit exists and is independent of the details of choosing the points z_j and ζ_j, where ζ_j is a point on the curve bewteen z_j and z_{j-1}.



The right-hand side of the above equation is called the contour (path) integral of f(z)

Analytic functions: Part Three

As an alternative, the contour may be defined by

$$\int_{c\ z_1}^{z_2} f(z)dz = \int_{c\ x_1 y_1}^{x_2 y_2} [u(x,y) + iv(x,y)][dx + idy]$$

$$= \int_{c\ x_1 y_1}^{x_2 y_2} [udx - vdy] + i \int_{c\ x_1 y_1}^{x_2 y_2} [vdx + udy]$$

with the path C specified. This reduces the complex integral to the complex sum of real integrals. It’s somewhat analogous to the case of the vector integral.

Analytic functions: Part Four

An important example $\int_{C} z^{n} dz$

where C is a circle of radius r>0 around the origin z=0 in the direction of counterclockwise.

In polar coordinates, we parameterize $z = re^{i\theta}$

and $dz = ire^{i\theta} d\theta$, and have

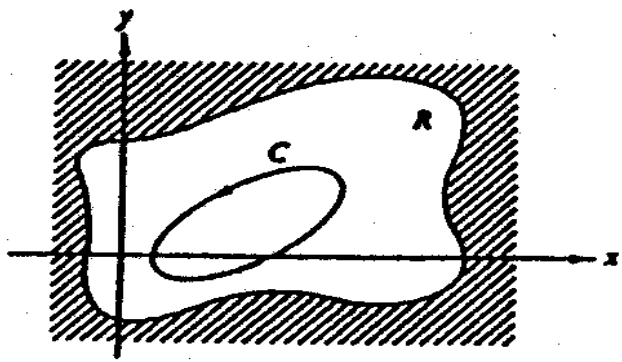
$$\frac{1}{2\pi i}\int_{C} z^{n} dz = \frac{r^{n+1}}{2\pi}\int_{0}^{2\pi} \exp[i(n+1)\theta]\, d\theta$$

$$= \begin{cases} 0 & \text{for } n \neq -1 \\ 1 & \text{for } n = -1 \end{cases}$$

Cauchy's integral Theorem: Part One

If a function f(z) is analytical (therefore single-valued) [and its partial derivatives are continuous] through some simply connected region **R**, for every closed path C in **R**,

$$\oint_c f(z)dz = 0$$



Stokes' theorem:

Proof: (under relatively restrictive condition: the partial derivative of u, v are continuous, which are actually not required but usually satisfied in physical problems)

$$\oint_c f(z)dz = \oint_c (udx - vdy) + i\oint_c (vdx + udy)$$

Cauchy's integral Theorem: Part Two

These two line integrals can be converted to surface integrals by Stokes' theorem

$$\oint_c \underline{A} \cdot d\underline{l} = \int_s \nabla \times \underline{A} \cdot d\underline{s}$$

$$\underline{A} = A_x \hat{x} + A_y \hat{y} \qquad ds = dxdy\hat{z}$$

$$\oint_c (A_x dx + A_y dy) = \oint_c \underline{A} \cdot d\underline{l} = \int_s \nabla \times \underline{A} \cdot d\underline{s} = \int \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy$$

For the real part, If we let u = A_x, and v = $-A_y$, then

$$\oint_c (udx - vdy) = -\int \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy$$

=0 [since C-R conditions $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$]

Cauchy's integral Theorem:Part Three

For the imaginary part, setting u = A_y and v = A_x, we have

$$\oint (v\,dx + u\,dy) = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy = 0$$

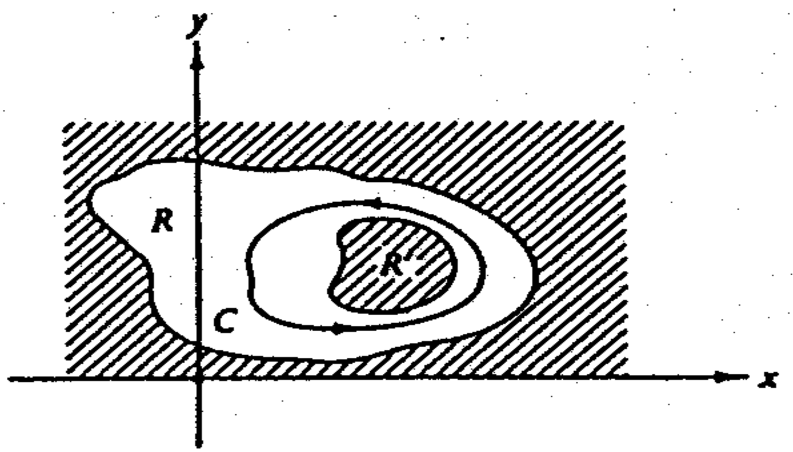
$$\oint \mathrm{f}(\mathrm{z})\,\mathrm{dz} = 0$$

The consequence of the theorem is that for analytic functions the line integral is a function only of its end points, independent of the path of integration,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)\,dz$$

Multiply Connected Regions: One

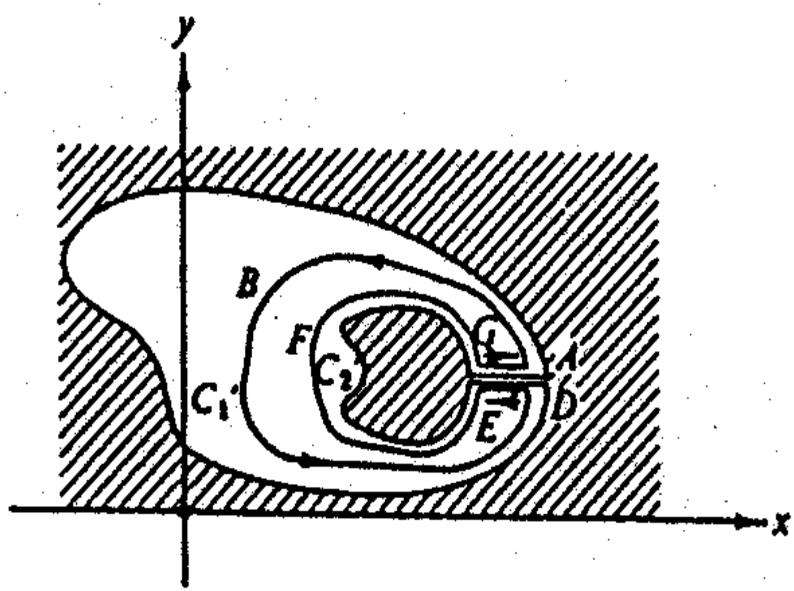
- The original statement of our theorem demanded a simply connected region. This restriction may easily be relaxed by the creation of a barrier, a contour line.
- Consider the multiply connected region of the figure below In which f(z) is not defined for the interior R′



Multiply Connected Regions: Two

Cauchy's integral theorem is not valid for the contour C, but we can construct a C′ for which the theorem holds. If line segments DE and GA arbitrarily close together, then

$$\int_G^A f(z)dz = -\int_D^E f(z)dz$$



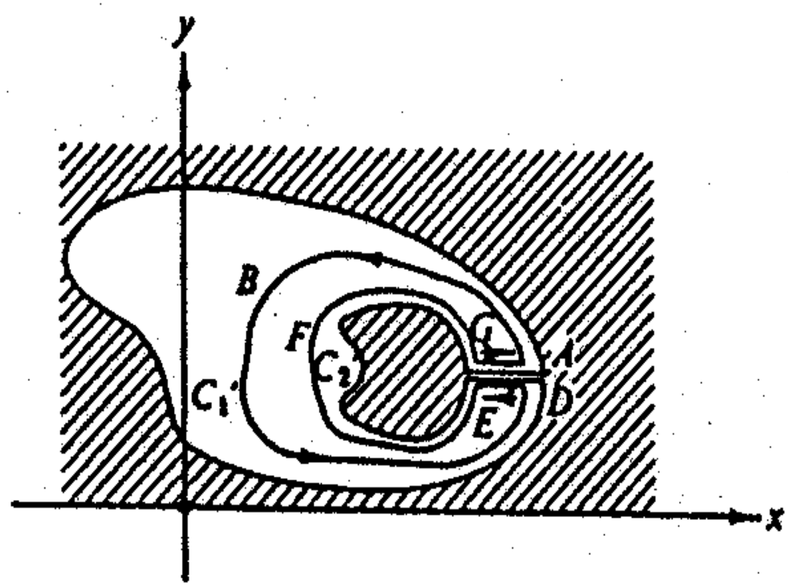
Multiply Connected Regions: Three

$$\oint_{\substack{C' \\ (ABDEFGA)}} f(z)dz = \left[\int_{ABD} + \int_{DE} + \int_{GA} + \int_{EFG}\right] f(z)dz$$

$$= \left[\int_{\mathrm{ABD}} + \int_{\mathrm{EFG}}\right] \mathrm{f}(\mathrm{z})\mathrm{dz} = 0$$

$$\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz$$

$$ABD \rightarrow C_1' \qquad EFG \rightarrow -C_2'$$



Cauchy’s Integral Formula: One

If f(z) is analytic on and within a closed contour C then

$$\oint_C \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0)$$

in which z_0 is some point in the interior region bounded by C. Note that here $z-z_0 \neq 0$ and the integral is well defined.

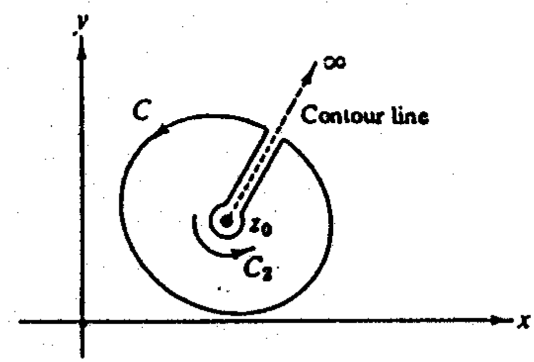
Although f(z) is assumed analytic, the integrand $(f(z)/z-z_0)$ is not analytic at $z=z_0$ unless $f(z_0)=0$. If the contour is deformed as in the figure on the next slide

Cauchy’s integral theorem applies.

Cauchy's Integral Formula: Two

So we have

$$\oint_C \frac{f(z)dz}{z-z_0} - \oint_{C_2} \frac{f(z)}{z-z_0}\,dz = 0$$



Let $z - z_0 = re^{i\theta}$, here r is small and will eventually be made to approach zero

$$\oint_{C_2} \frac{f(z)dz}{z-z_0}\,dz = \oint_{C_2} \frac{f\left(z_0 + re^{i\theta}\right)}{re^{i\theta}} rie^{i\theta}\,d\theta \quad = if(z_0)\oint_{C_2 (r \to 0)} d\theta = 2\pi i f(z_0)$$

Cauchy's Integral Formula: Three

Here is a remarkable result. The value of an analytic function is given at an interior point at $z=z_0$ once the values on the boundary C are specified.

What happens if z_0 is exterior to C?

In this case the entire integral is analytic on and within C, so the integral vanishes.

$$\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-z_0} = \begin{cases} f(z_0), & z_0 \text{ interior} \\ 0, & z_0 \text{ exterior} \end{cases}$$

Cauchy’s Integral Formula: Four

Cauchy’s integral formula may be used to obtain an expression for the derivation of f(z)

$$f'(z_0) = \frac{d}{dz_0}\left(\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z - z_0}\right)$$

$$= \frac{1}{2\pi i}\oint f(z)dz\,\frac{d}{dz_0}\left(\frac{1}{z - z_0}\right) = \frac{1}{2\pi i}\oint \frac{f(z)dz}{(z - z_0)^2}$$

Moreover, for the n-th order of derivative

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)dz}{(z - z_0)^{n+1}}$$

Cauchy's Integral Formula: Five

We now see that, the requirement that f(z) be analytic not only guarantees a first derivative but derivatives of all orders as well! The derivatives of f(z) are automatically analytic. Here, it is worth to indicate that the converse of Cauchy's integral theorem holds as well

Morera's theorem:

If a function f(z) is continuous in a simply connected region R and $\oint_C f(z)dz = 0$ for every closed C within R, then f(z) is $\therefore$ analytic throught R (see the text book).

Liouville's Theorem: Part One

Liouville's theorem: If f(z) is analytic and bounded in the complex plane, it is a constant.

Proof: For any z_0, construct a circle of radius R around z_0,

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \oint_R \frac{f(z)dz}{(z - z_0)^2} \right| \leq \frac{M}{2\pi} \frac{2\pi R}{R^2} = \frac{M}{R}$$

Liouville's Theorem: Part Two

Since R is arbitrary, let $R \to \infty$, we have

$$f'(z) = 0, \text{i.e}, f(z) = \text{const.}$$

Conversely, the slightest deviation of an analytic function from a constant value implies that there must be at least one singularity somewhere in the infinite complex plane. Apart from the trivial constant functions, then, singularities are a fact of life, and we must learn to live with them, and to use them further.

Laurent Series: Part One

Taylor Expansion

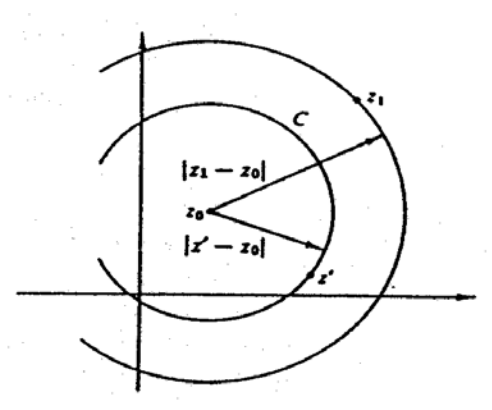
Suppose we are trying to expand f(z) about z=z0, i.e., $f(z) = \sum_{n-0}^{\infty} a_n (z - z_0)^n$

and we have $z=z_1$ as the nearest point for which f(z) is not analytic. We construct a circle C centered at $z=z_0$ with radius $|z' - z_0| < |z_1 - z_0|$

From the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{z'-z} = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]}$$



Laurent Series: Part Two

Here z' is a point on C and z is any point interior to C. For $|t| < 1$, we note the identity

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{n=0}^{\infty} t^n$$

So we may write

$$f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n f(z') dz'}{(z' - z_0)^{n+1}}$$

which is our desired Taylor expansion, just as for real variable power series, this expansion is unique for a given z_0.

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z') dz'}{(z' - z_0)^{n+1}} \quad = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

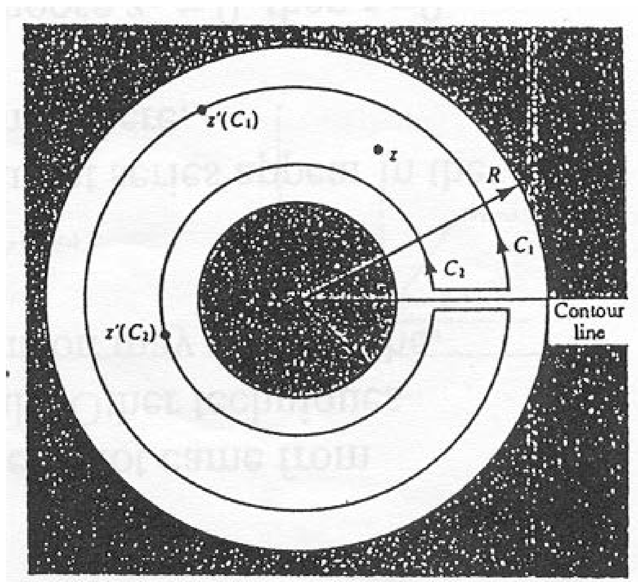
From the binomial expansion of $g(z) = (z - x_0)^n$ for integer n, it is easy to see, for real x_0

Laurent Series: Part Three

We frequently encounter functions that are analytic in annular region

Drawing an imaginary contour line to convert our region into a simply connected region, we apply Cauchy's integral formula for C_2 and C_1, with radii r_2 and r_1, and obtain

$$f(z) = \frac{1}{2\pi i}\left[\oint_{C_1} - \oint_{C_2}\right]\frac{f(z')dz'}{z' - z}$$



Laurent Series: Part Four

We let $r_2 \to r$ and $r_1 \to R$, so for C_1, $|z' - z_0| > |z - z_0|$ while for C_2, $|z' - z_0| < |z - z_0|$.
We expand two denominators and we get:

$$f(z) = \frac{1}{2\pi i}\left\{ \oint_{C_1} \frac{f(z')dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]} + \oint_{C_2} \frac{f(z')dz'}{(z-z_0)[1-(z'-z_0)/(z-z_0)]} \right\}$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1} \frac{f(z')dz'}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}} \oint_{C_2} (z'-z_0)^n f(z')dz'$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad \text{(Laurent Series)}$$

Laurent Series: Part Five

Drawing an imaginary contour line to convert our region into a simply connected region, we apply Cauchy's integral formula for C2 and C1, with radii r2 and r1, and obtain

$$f(z) = \frac{1}{2\pi i}\left[\oint_{C_1} - \oint_{C_2}\right]\frac{f(z')dz'}{z'-z}$$

We let $r_2 \to r$ and $r_1 \to R$, so for C_1, $|z'-z_0| > |z-z_0|$ while for C_2, $|z'-z_0| < |z-z_0|$.
We expand two denominators as we did before

Laurent Series: Part Six

Where:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}}$$

Here C may be any contour with the annular region r < $|z - z_0|$ < R encircling z_0 once in a counterclockwise sense.

Laurent Series need not to come from evaluation of contour integrals. Other techniques such as ordinary series expansion may provide the coefficients.

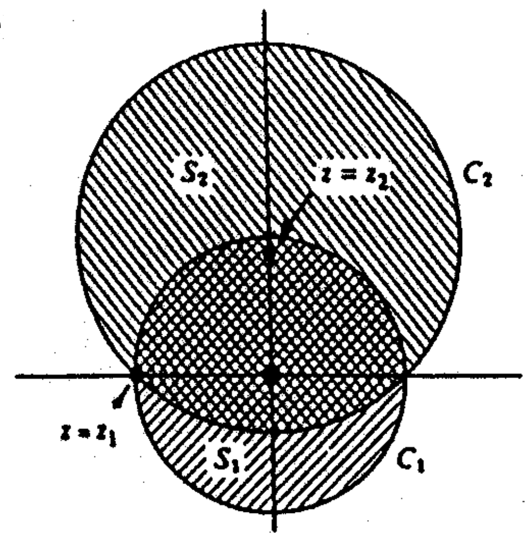
Analytic Continuation: Part One

For example $f(z) = 1/(1+z)$

which has a simple pole at z = -1 and is analytic
elsewhere. For |z| < 1, the geometric series e
while expanding it about z=i leads to f_2,

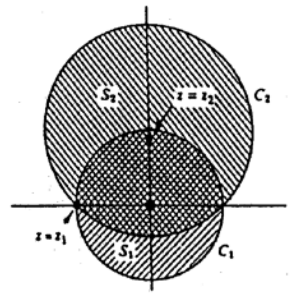
$$f(z) = \frac{1}{1+z}; \qquad f_1 = \sum_{n=0}^{\infty} (-z)^n; \qquad f_2 = \frac{1}{1+i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{z+i} \right)^n$$

$$\frac{1}{1+z} = 1 - z + z^2 + \cdots = \sum_{n=0}^{\infty} (-z)^n$$



Analytic Continuation: Part Two

Suppose we expand it about z = i, so that



$$f(z) = \frac{1}{1 + i + (z - i)} = \frac{1}{(1+i)[1 + (z-i)/(1+i)]}$$

$$= \frac{1}{1+i}\left[1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} + \cdots\right]$$

converges for $|z - i| < |1 + i| = \sqrt{2}$ (Fig.1.10)

The above three equations are different representations of the same function. Each representation has its own domain of convergence.

If two analytic functions coincide in any region, such as the overlap of s1 and s2, of coincide on any line segment, they are the same function in the sense that they will coincide everywhere as long as they are well-defined.